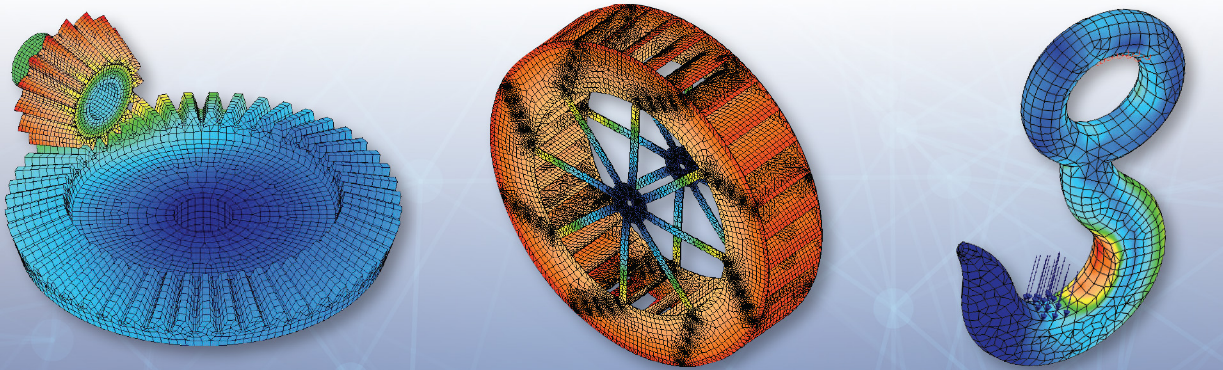
SI EDITION

SIXTH EDITION

A FIRST COURSE IN THE FINITE ELEMENT METHOD



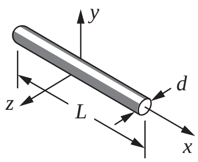
Daryl L. Logan

CONVERSION FACTORS U.S. Customary Units to SI Units

Quantity Converted from U.S. Customary	To	SI Equivalent
(Acceleration)		
1 foot/second^2 (ft/s^2)	meter/second^2 (m/s^2)	0.3048 m/s^2
1 inch/second^2 (in./s^2)	meter/second^2 (m/s^2)	0.0254 m/s^2
(Area)		
1 foot^2 (ft^2)	meter^2 (m^2)	0.0929 m^2
1 inch^2 (in.^{2})	meter^2 (m^2)	645.2 mm^2
(Density, mass)		
1 pound mass/inch^3 (lbm/in.^{3})	kilogram/meter^3 (kg/m^3)	27.68 Mg/m^3
1 pound mass/foot^3 (lbm/ft^3)	kilogram/meter^3 (kg/m^3)	16.02 kg/m^3
(Energy, Work)		
1 British thermal unit (BTU)	Joule (J)	1055 J
1 foot-pound force (ft-lb)	Joule (J)	1.356 J
1 kilowatt-hour	Joule (J)	3.60×10^6 J
(Force)		
1 kip (1000 lb)	Newton (N)	4.448 kN
1 pound force (lb)	Newton (N)	4.448 N
(Length)		
1 foot (ft)	meter (m)	0.3048 m
1 inch (in.)	meter (m)	25.4 mm
1 mile (mi), (U.S. statute)	meter (m)	1.609 km
1 mile (mi), (international nautical)	meter (m)	1.852 km
(Mass)		
1 pound mass (lbm)	kilogram (kg)	0.4536 kg
1 slug (lb-sec^2/ft)	kilogram (kg)	14.59 kg
1 metric ton (2000 lbm)	kilogram (kg)	907.2 kg
(Moment of force)		
1 pound-foot (lb·ft)	Newton-meter (N·m)	1.356 N·m
1 pound-inch (lb·in.)	Newton-meter (N·m)	0.1130 N·m
(Moment of inertia of an area)		
1 inch^4	meter^4 (m^4)	0.4162×10^{-6} m^4
(Moment of inertia of a mass)		
1 pound-foot-second^2(lb·ft·s^2)	kilogram-meter^2 (kg·m^2)	1.356 kg·m^2
(Momentum, linear)		
1 pound-second (lb·s)	kilogram-meter/second (kg·m/s)	4.448 N·s
(Momentum, angular)		
pound-foot-second (lb·ft·s)	Newton-meter-second (N·m·s)	1.356 N·m·s

PROPERTIES OF SOLIDS **Notes:** ρ = **mass density,** m = **mass,** I = **mass moment of inertia.**

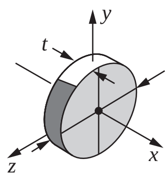
1. Slender Rod



$$m = \frac{\pi d^2 L \rho}{4}$$

$$I_y = I_z = \frac{mL^2}{12}$$

2. Thin Disk

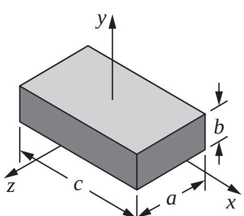


$$m = \frac{\pi d^2 t \rho}{4}$$

$$I_x = \frac{md^2}{8}$$

$$I_y = I_z = \frac{md^2}{16}$$

3. Rectangular Prism



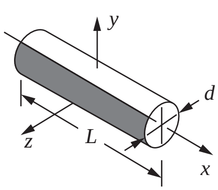
$$m = abc\rho$$

$$I_x = \frac{m}{12}(a^2 + b^2)$$

$$I_y = \frac{m}{12}(a^2 + c^2)$$

$$I_z = \frac{m}{12}(b^2 + c^2)$$

4. Circular Cylinder

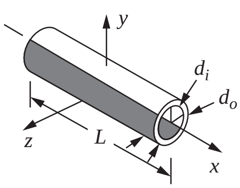


$$m = \frac{\pi d^2 L \rho}{4}$$

$$I_x = \frac{md^2}{8}$$

$$I_y = I_z = \frac{m}{48}(3d^2 + 4L^2)$$

5. Hollow Cylinder



$$m = \frac{\pi L \rho}{4}(d_o^2 - d_i^2)$$

$$I_x = \frac{m}{8}(d_o^2 + d_i^2)$$

$$I_y = I_z = \frac{m}{48}(3d_o^2 + 3d_i^2 + 4L^2)$$

CONVERSION FACTORS U.S. Customary Units to SI Units (*Continued*)

Quantity Converted from U.S. Customary	To	SI Equivalent
(Power)		
1 foot-pound/second (ft·lb/s)	Watt (W)	1.356 W
1 horsepower (550 ft·lb/s)	Watt (W)	745.7 W
(Pressure, stress)		
1 atmosphere (std)(14.7.lb/in.$^{2)}$	Newton/meter^2 (N/m^2 or Pa)	101.3 kPa
1 pound/foot^2 (lb/ft^2)	Newton/meter^2 (N/m^2 or Pa)	47.88 Pa
1 pound/inch^2 (lb/in.^{2} or psi)	Newton/meter^2 (N/m^2 or Pa)	6.895 kPa
1 kip/inch^2(ksi)	Newton/meter^2 (N/m^2 or Pa)	6.895 MPa
(Spring constant)		
1 pound/inch (lb/in.)	Newton/meter (N/m)	175.1 N/m
(Temperature)		
T(°F) = 1.8T(°C) + 32		
(Velocity)		
1 foot/second (ft/s)	meter/second (m/s)	0.3048 m/s
1 knot (nautical mi/h)	meter/second (m/s)	0.5144 m/s
1 mile/hour (mi/h)	meter/second (m/s)	0.4470 m/s
1 mile/hour (mi/h)	kilometer/hour (km/h)	1.609 km/h
(Volume)		
1 foot^3 (ft^3)	meter^3 (m^3)	0.02832 m^3
1 inch^3 (in.^{3})	meter^3 (m^3)	16.39×10^{-6} m^3

A First Course in the Finite Element Method

SIXTH EDITION, SI

Daryl L. Logan

University of Wisconsin–Platteville

CENGAGE Learning™

Australia • Brazil • Mexico • Singapore • United Kingdom • United States

CENGAGE Learning™

***A First Course in the Finite Element Method*, Sixth Edition, SI**
Daryl L. Logan
SI Edition Prepared by Qaboos Imran

Product Director, Global Engineering: Timothy L. Anderson

Senior Content Developer: Mona Zeftel

Associate Media Content Developer: Ashley Kaupert

Product Assistant: Teresa Versaggi

Marketing Manager: Kristin Stine

Director, Content and Media Production: Sharon L. Smith

Content Project Manager: Tim Bailey

Production Service: RPK Editorial Services, Inc.

Copyeditor: Harlan James

Proofreader: Martha McMaster

Indexer: Shelly Gerger-Knechtl

Compositor: SPI-Global

Senior Art Director: Michelle Kunkler

Cover and Internal Designer: Harasymczuk

Cover Image: Kevin Roholt, Eric Simon, John Stec using Autodesk Mechanical Simulation 2014 and 2015

Intellectual Property
Analyst: Christine Myaskovsky
Project Manager: Sarah Shainwald

Text and Image Permissions Researcher: Kristiina Paul

Manufacturing Planner: Doug Wilke

Library of Congress Control Number: 2016940946

ISBN-13: 978-1-305-63734-4

Cengage Learning
20 Channel Center Street
Boston MA 02210
USA

Printed in the United States of America

Print Number: 01 Print Year: 2016

CONTENTS

Preface to the SI Edition xii
Preface xiii
Acknowledgments xv
Notation xvi

1 Introduction 1

Chapter Objectives 1
Prologue 1
1.1 Brief History 3
1.2 Introduction to Matrix Notation 4
1.3 Role of the Computer 6
1.4 General Steps of the Finite Element Method 7
1.5 Applications of the Finite Element Method 15
1.6 Advantages of the Finite Element Method 21
1.7 Computer Programs for the Finite Element Method 25
Reference 27
Problems 30

2 Introduction to the Stiffness (Displacement) Method 31

Chapter Objectives 31
Introduction 31
2.1 Definition of the Stiffness Matrix 32
2.2 Derivation of the Stiffness Matrix for a Spring Element 32
2.3 Example of a Spring Assemblage 36
2.4 Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method) 38
2.5 Boundary Conditions 40
2.6 Potential Energy Approach to Derive Spring Element Equations 55
Summary Equations 65
References 66
Problems 66

3 Development of Truss Equations 72

Chapter Objectives 72
Introduction 72
3.1 Derivation of the Stiffness Matrix for a Bar Element in Local Coordinates 73
3.2 Selecting a Displacement Function in Step 2 of the Derivation of Stiffness Matrix for the One-Dimensional Bar Element 78
3.3 Transformation of Vectors in Two Dimensions 82
3.4 Global Stiffness Matrix for Bar Arbitrarily Oriented in the Plane 84
3.5 Computation of Stress for a Bar in the $x - y$ Plane 89
3.6 Solution of a Plane Truss 91
3.7 Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space 100
3.8 Use of Symmetry in Structures 109
3.9 Inclined, or Skewed, Supports 112
3.10 Potential Energy Approach to Derive Bar Element Equations 121
3.11 Comparison of Finite Element Solution to Exact Solution for Bar 132
3.12 Galerkin's Residual Method and Its Use to Derive the One-Dimensional Bar Element Equations 136
3.13 Other Residual Methods and Their Application to a One-Dimensional Bar Problem 139
3.14 Flowchart for Solution of Three-Dimensional Truss Problems 143
3.15 Computer Program Assisted Step-by-Step Solution for Truss Problem 144
Summary Equations 146
References 147
Problems 147

4 Development of Beam Equations 169

Chapter Objectives 169
Introduction 169
4.1 Beam Stiffness 170
4.2 Example of Assemblage of Beam Stiffness Matrices 180
4.3 Examples of Beam Analysis Using the Direct Stiffness Method 182
4.4 Distributed Loading 195

4.5 Comparison of the Finite Element Solution to the Exact Solution for a Beam 208
4.6 Beam Element with Nodal Hinge 214
4.7 Potential Energy Approach to Derive Beam Element Equations 222
4.8 Galerkin's Method for Deriving Beam Element Equations 225
Summary Equations 227
References 228
Problems 229

5 Frame and Grid Equations 239

Chapter Objectives 239
Introduction 239
5.1 Two-Dimensional Arbitrarily Oriented Beam Element 239
5.2 Rigid Plane Frame Examples 243
5.3 Inclined or Skewed Supports—Frame Element 261
5.4 Grid Equations 262
5.5 Beam Element Arbitrarily Oriented in Space 280
5.6 Concept of Substructure Analysis 295
Summary Equations 300
References 302
Problems 303

6 Development of the Plane Stress and Plane Strain Stiffness Equations 337

Chapter Objectives 337
Introduction 337
6.1 Basic Concepts of Plane Stress and Plane Strain 338
6.2 Derivation of the Constant-Strain Triangular Element Stiffness Matrix and Equations 342
6.3 Treatment of Body and Surface Forces 357
6.4 Explicit Expression for the Constant-Strain Triangle Stiffness Matrix 362
6.5 Finite Element Solution of a Plane Stress Problem 363
6.6 Rectangular Plane Element (Bilinear Rectangle, Q4) 374
Summary Equations 379
References 384
Problems 384

7 **Practical Considerations in Modeling; Interpreting Results; and Examples of Plane Stress/Strain Analysis 391**

Chapter Objectives 391
Introduction 391
7.1 Finite Element Modeling 392
7.2 Equilibrium and Compatibility of Finite Element Results 405
7.3 Convergence of Solution and Mesh Refinement 408
7.4 Interpretation of Stresses 411
7.5 Flowchart for the Solution of Plane Stress/Strain Problems 413
7.6 Computer Program–Assisted Step-by-Step Solution, Other Models, and Results for Plane Stress/Strain Problems 414
References 420
Problems 421

8 **Development of the Linear-Strain Triangle Equations 437**

Chapter Objectives 437
Introduction 437
8.1 Derivation of the Linear-Strain Triangular Element Stiffness Matrix and Equations 437
8.2 Example LST Stiffness Determination 442
8.3 Comparison of Elements 444
Summary Equations 447
References 448
Problems 448

9 **Axisymmetric Elements 451**

Chapter Objectives 451
Introduction 451
9.1 Derivation of the Stiffness Matrix 451
9.2 Solution of an Axisymmetric Pressure Vessel 462
9.3 Applications of Axisymmetric Elements 468
Summary Equations 473
References 475
Problems 475

10 **Isoparametric Formulation 486**

Chapter Objectives 486
Introduction 486

10.1 Isoparametric Formulation of the Bar Element Stiffness Matrix 487
10.2 Isoparametric Formulation of the Plane Quadrilateral (Q4) Element Stiffness Matrix 492
10.3 Newton-Cotes and Gaussian Quadrature 503
10.4 Evaluation of the Stiffness Matrix and Stress Matrix by Gaussian Quadrature 509
10.5 Higher-Order Shape Functions (Including Q6, Q8, Q9, and Q12 Elements) 515
Summary Equations 526
References 530
Problems 530

11 Three-Dimensional Stress Analysis 536

Chapter Objectives 536
Introduction 536
11.1 Three-Dimensional Stress and Strain 537
11.2 Tetrahedral Element 539
11.3 Isoparametric Formulation and Hexahedral Element 547
Summary Equations 555
References 558
Problems 558

12 Plate Bending Element 572

Chapter Objectives 572
Introduction 572
12.1 Basic Concepts of Plate Bending 572
12.2 Derivation of a Plate Bending Element Stiffness Matrix and Equations 577
12.3 Some Plate Element Numerical Comparisons 582
12.4 Computer Solutions for Plate Bending Problems 584
Summary Equations 588
References 590
Problems 591

13 Heat Transfer and Mass Transport 599

Chapter Objectives 599
Introduction 599
13.1 Derivation of the Basic Differential Equation 601
13.2 Heat Transfer with Convection 604

13.3 Typical Units; Thermal Conductivities, *K*; and Heat Transfer Coefficients, *h* 605
13.4 One-Dimensional Finite Element Formulation Using a Variational Method 607
13.5 Two-Dimensional Finite Element Formulation 626
13.6 Line or Point Sources 636
13.7 Three-Dimensional Heat Transfer by the Finite Element Method 639
13.8 One-Dimensional Heat Transfer with Mass Transport 641
13.9 Finite Element Formulation of Heat Transfer with Mass Transport by Galerkin's Method 642
13.10 Flowchart and Examples of a Heat Transfer Program 646
Summary Equations 651
References 654
Problems 655

14 Fluid Flow in Porous Media and through Hydraulic Networks; and Electrical Networks and Electrostatics 673

Chapter Objectives 673
Introduction 673
14.1 Derivation of the Basic Differential Equations 674
14.2 One-Dimensional Finite Element Formulation 678
14.3 Two-Dimensional Finite Element Formulation 691
14.4 Flowchart and Example of a Fluid-Flow Program 696
14.5 Electrical Networks 697
14.6 Electrostatics 701
Summary Equations 715
References 719
Problems 720

15 Thermal Stress 727

Chapter Objectives 727
Introduction 727
15.1 Formulation of the Thermal Stress Problem and Examples 727
Summary Equations 752
Reference 753
Problems 754

16 Structural Dynamics and Time-Dependent Heat Transfer 761

Chapter Objectives 761
Introduction 761
16.1 Dynamics of a Spring-Mass System 762
16.2 Direct Derivation of the Bar Element Equations 764
16.3 Numerical Integration in Time 768
16.4 Natural Frequencies of a One-Dimensional Bar 780
16.5 Time-Dependent One-Dimensional Bar Analysis 784
16.6 Beam Element Mass Matrices and Natural Frequencies 789
16.7 Truss, Plane Frame, Plane Stress, Plane Strain, Axisymmetric, and Solid Element Mass Matrices 796
16.8 Time-Dependent Heat Transfer 801
16.9 Computer Program Example Solutions for Structural Dynamics 808
Summary Equations 817
References 821
Problems 822

Appendix A Matrix Algebra 827

Appendix B Methods for Solution of Simultaneous Linear Equations 843

Appendix C Equations from Elasticity Theory 865

Appendix D Equivalent Nodal Forces 873

Appendix E Principle of Virtual Work 876

Appendix F Geometric Properties of Structural Steel Wide-Flange Sections (W Shapes) 880

Answers to Selected Problems 908

Index 938

PREFACE TO THE SI EDITION

This edition of *A First Course in the Finite Element Method, Sixth Edition* has been adapted to incorporate the International System of Units (*Le Système International d'Unités* or SI) throughout the book.

Le Système International d'Unités

The United States Customary System (USCS) of units uses FPS (foot–pound–second) units (also called English or Imperial units). SI units are primarily the units of the MKS (meter–kilogram–second) system. However, CGS (centimeter–gram–second) units are often accepted as SI units, especially in textbooks.

Using SI Units in this Book

In this book, we have used both MKS and CGS units. USCS (U.S. Customary Units) or FPS (foot-pound-second) units used in the US Edition of the book have been converted to SI units throughout the text and problems. However, in case of data sourced from handbooks, government standards, and product manuals, it is not only extremely difficult to convert all values to SI, it also encroaches upon the intellectual property of the source. Some data in figures, tables, and references, therefore, remains in FPS units. For readers unfamiliar with the relationship between the USCS and the SI systems, a conversion table has been provided inside the front cover.

To solve problems that require the use of sourced data, the sourced values can be converted from FPS units to SI units just before they are to be used in a calculation. To obtain standardized quantities and manufacturers' data in SI units, readers may contact the appropriate government agencies or authorities in their regions.

Instructor Resources

The Instructors' Solution Manual in SI units is available through your Sales Representative or online through the book website at http://login.cengage.com. A digital version of the ISM, Lecture Note PowerPoint slides for the SI text, as well as other resources are available for instructors registering on the book website.

Feedback from users of this SI Edition will be greatly appreciated and will help us improve subsequent editions.

PREFACE

Features and Approach

The purpose of this sixth edition is to provide an introductory approach to the finite element method that can be understood by both undergraduate and graduate students without the usual prerequisites (such as structural analysis and upper level calculus) required by many available texts in this area. The book is written primarily as a basic learning tool for the undergraduate student in civil and mechanical engineering whose main interest is in stress analysis and heat transfer, although material on fluid flow in porous media and through hydraulic networks and electrical networks and electrostatics is also included. The concepts are presented in sufficiently simple form with numerous example problems logically placed throughout the book, so that the book serves as a valuable learning aid for students with other backgrounds, as well as for practicing engineers. The text is geared toward those who want to apply the finite element method to solve practical physical problems.

General principles are presented for each topic, followed by traditional applications of these principles, including longhand solutions, which are in turn followed by computer applications where relevant. The approach is taken to illustrate concepts used for computer analysis of large-scale problems.

The book proceeds from basic to advanced topics and can be suitably used in a two-course sequence. Topics include basic treatments of (1) simple springs and bars, leading to two- and three-dimensional truss analysis; (2) beam bending, leading to plane frame, grid, and space frame analysis; (3) elementary plane stress/strain elements, leading to more advanced plane stress/strain elements and applications to more complex plane stress/strain analysis; (4) axisymmetric stress analysis; (5) isoparametric formulation of the finite element method; (6) three-dimensional stress analysis; (7) plate bending analysis; (8) heat transfer and fluid mass transport; (9) basic fluid flow through porous media and around solid bodies, hydraulic networks, electric networks, and electrostatics analysis; (10) thermal stress analysis; and (11) time-dependent stress and heat transfer.

Additional features include how to handle inclined or skewed boundary conditions, beam element with nodal hinge, the concept of substructure, the patch test, and practical considerations in modeling and interpreting results.

The direct approach, the principle of minimum potential energy, and Galerkin's residual method are introduced at various stages, as required, to develop the equations needed for analysis.

Appendices provide material on the following topics: (A) basic matrix algebra used throughout the text; (B) solution methods for simultaneous equations; (C) basic theory of elasticity; (D) work-equivalent nodal forces; (E) the principle of virtual work; and (F) properties of structural steel shapes.

More than 100 solved examples appear throughout the text. Most of these examples are solved "longhand" to illustrate the concepts. More than 570 end-of-chapter problems are provided to reinforce concepts. The answers to many problems are included in the back of the book to aid those wanting to verify their work. Those end-of-chapter problems to be solved using a computer program are marked with a computer symbol.

Additional Features

Additional features of this edition include updated notation used by most engineering instructors, chapter objectives at the start of each chapter to help students identify what content is most important to focus on and retain summary equations for handy use at the end of each chapter, additional information on modeling, and more comparisons of finite element solutions to analytical solutions.

New Features

Over 140 new problems for solution have been included, and additional design-type problems have been added to chapters 3, 5, 7, 11, and 12. Additional real-world applications from industry have been added to enhance student understanding and reinforce concepts. New space frames, solid-model-type examples, and problems for solution have been added. New examples from other fields now demonstrate how students can use the Finite Element Method to solve problems in a variety of engineering and mathematical physics areas. As in the 5th edition, this edition deliberately leaves out consideration of special purpose computer programs and suggests that instructors choose a program they are familiar with to integrate into their finite element course.

Resources for Instructors

To access instructor resources, including a secure, downloadable Instructor's Solution Manual and Lecture Note PowerPoint Slides, please visit our Instructor Resource Center at http://sso.cengage/com.

MindTap Online Course

This textbook is also available online through Cengage Learning's MindTap, a personalized learning program. Students who purchase the MindTap have access to the book's multimedia-rich electronic Reader and are able to complete homework and assessment material online, on their desktops, laptops, or iPads. The new MindTap Mobile App makes it easy for students to study anywhere, anytime. Instructors who use a Learning Management System (such as Blackboard or Moodle) for tracking course content, assignments, and grading, can seamlessly access the MindTap suite of content and assessments for this course.

With MindTap, instructors can:

- Personalize the Learning Path to match the course syllabus by rearranging content or appending original material to the online content
- Connect a Learning Management System portal to the online course and Reader
- Highlight and annotate the digital textbook, then share their notes with students.
- Customize online assessments and assignments
- Track student engagement, progress and comprehension
- Promote student success through interactivity, multimedia, and exercises

Additionally, students can listen to the text through ReadSpeaker, take notes in the digital Reader, study from and create their own Flashcards, highlight content for easy reference, and check their understanding of the material through practice quizzes and automatically-graded homework.

Suggested Topics

Following is an outline of suggested topics for a first course (approximately 44 lectures, 52 minutes each) in which this textbook is used.

Topic	Number of Lectures
Appendix A	1
Appendix B	1
Chapter 1	2
Chapter 2	3
Chapter 3, Sections 3.1–3.11, 3.14 and 3.15	5
Exam 1	1
Chapter 4, Sections 4.1–4.6	4
Chapter 5, Sections 5.1–5.3, 5.5	4
Chapter 6	4
Chapter 7	3
Exam 2	1
Chapter 9	2
Chapter 10	4
Chapter 11	3
Chapter 13, Sections 13.1–13.7	5
Chapter 15	3
Exam 3	1

Acknowledgments

I express my deepest appreciation to the staff at Cengage Learning especially Tim Anderson, Publisher; Mona Zeftel, Content Developer and Teresa Versaggi, Product Assistant; and to Rose Kernan of RPK Editorial Services for their assistance in producing this new edition.

I am grateful to Dr. Ted Belytschko for his excellent teaching of the finite element method, which aided me in writing this text. Also I want to thank the many students at the University for their suggestions on ways to make topics easier to understand. These suggestions have been incorporated into this edition as well.

I also wish to thank the reviewers for this sixth edition who include Raghu Agarwal, San Jose State University; Mohammad Alimi, California State University, Fresno; K. Chandrashekhara, Missouri University of Science and Technology; Howie Fang, University of North Carolina, Charlotte; Winfred Foster, Auburn University; Kenneth Miller, St. Cloud State University; Ronald Rorrer, University of Colorado, Denver; and Charles Yang, Wichita State University.

I thank many students at the University of Wisconsin-Platteville (UWP), whose names are credited throughout the book, for contributing various two-and three-dimensional models from the finite element course. Thank you also to UWP graduate student William Gobeli for Table 11-2. Also special thanks to Andrew Heckman, an alum of UWP and Design Engineer at Seagraves Fire Apparatus for permission to use Figure 11-10 and to Mr. Yousif Omer, Structural Engineer at John Deere Dubuque Works for allowing permission to use Figure 1-11. I want to thank the people at Autodesk for their contribution of Figure 9-2b. Finally, I want to thank Ioan Giosan, Senior Design Engineer at Valmont West Coast Engineering for allowing permission to use Figures 1-12 and 1-13.

NOTATION

English Symbols

a_i	generalized coordinates (coefficients used to express displacement in general form)
A	cross-sectional area
$[B]$	matrix relating strains to nodal displacements or relating temperature gradient to nodal temperatures
c	specific heat of a material
$[C']$	matrix relating stresses to nodal displacements
C	direction cosine in two dimensions
C_x, C_y, C_z	direction cosines in three dimensions
$\{d\}$	element and structure nodal displacement matrix, both in global coordinates
$\{d'\}$	local-coordinate element nodal displacement matrix
D	bending rigidity of a plate
$[D]$	matrix relating stresses to strains
$[D']$	operator matrix given by Eq. (10.2.16)
e	exponential function
E	modulus of elasticity
$\{f\}$	global-coordinate nodal force matrix
$\{f'\}$	local-coordinate element nodal force matrix
$\{f_b\}$	body force matrix
$\{f_h\}$	heat transfer force matrix
$\{f_q\}$	heat flux force matrix
$\{f_Q\}$	heat source force matrix
$\{f_s\}$	surface force matrix
$\{F\}$	global-coordinate structure force matrix
$\{F_c\}$	condensed force matrix
$\{F_i\}$	global nodal forces
$\{F_0\}$	equivalent force matrix
$\{g\}$	temperature gradient matrix or hydraulic gradient matrix
G	shear modulus
h	heat-transfer (or convection) coefficient
i, j, m	nodes of a triangular element
I	principal moment of inertia
$[J]$	Jacobian matrix
k	spring stiffness
$[k]$	global-coordinate element stiffness or conduction matrix
$[k_c]$	condensed stiffness matrix, and conduction part of the stiffness matrix in heat-transfer problems
$[k']$	local-coordinate element stiffness matrix
$[k_h]$	convective part of the stiffness matrix in heat-transfer problems
$[K]$	global-coordinate structure stiffness matrix
K_{xx}, K_{yy}	thermal conductivities (or permeabilities, for fluid mechanics) in the x and y directions, respectively
L	length of a bar or beam element

m	maximum difference in node numbers in an element
$m(x)$	general moment expression
m_x, m_y, m_{xy}	moments in a plate
$[m']$, $[m]$	local element mass matrix
$[m_i']$	local nodal moments
$[M]$	global mass matrix
$[\mathrm{M}^*]$	matrix used to relate displacements to generalized coordinates for a linear-strain triangle formulation
$[M']$	matrix used to relate strains to generalized coordinates for a linear-strain triangle formulation
n_b	bandwidth of a structure
n_d	number of degrees of freedom per node
$[N]$	shape (interpolation or basis) function matrix
N_i	shape functions
p	surface pressure (or nodal heads in fluid mechanics)
p_r, p_z	radial and axial (longitudinal) pressures, respectively
P	concentrated load
$[P']$	concentrated local force matrix
q	heat flow (flux) per unit area or distributed loading on a plate
$\overline{q}$	rate of heat flow
q^*	heat flow per unit area on a boundary surface
Q	heat source generated per unit volume or internal fluid source
Q^*	line or point heat source
Q_x, Q_y	transverse shear line loads on a plate
r, θ, z	radial, circumferential, and axial coordinates, respectively
R	residual in Galerkin's integral
R_b	body force in the radial direction
R_{ix}, R_{iy}	nodal reactions in x and y directions, respectively
s, t, z'	natural coordinates attached to isoparametric element
S	surface area
t	thickness of a plane element or a plate element
t_i, t_j, t_m	nodal temperatures of a triangular element
T	temperature function
T_∞	free-stream temperature
$[T]$	displacement, force, and stiffness transformation matrix
$[T_i]$	surface traction matrix in the i direction
u, v, w	displacement functions in the x, y, and z directions, respectively
u_i, v_i, w_i	x, y, and z displacements at node i, respectively
U	strain energy
ΔU	change in stored energy
v	velocity of fluid flow
V	shear force in a beam
w	distributed loading on a beam or along an edge of a plane element
W	work
x_i, y_i, z_i	nodal coordinates in the x, y, and z directions, respectively
x', y', z'	local element coordinate axes
x, y, z	structure global or reference coordinate axes
$[X]$	body force matrix
X_b, Y_b	body forces in the x and y directions, respectively
Z_b	body force in longitudinal direction (axisymmetric case) or in the z direction (three-dimensional case)

Greek Symbols

α	coefficient of thermal expansion
$\alpha_i, \beta_i, \gamma_i, \delta_i$	used to express the shape functions defined by Eq. (6.2.10) and Eqs. (11.2.5) through (11.2.8)
δ	spring or bar deformation
ϵ	normal strain
$\{\varepsilon_T\}$	thermal strain matrix
k_x, k_y, k_{xy}	curvatures in plate bending
ν	Poisson's ratio
ϕ_i	nodal angle of rotation or slope in a beam element
π_h	functional for heat-transfer problem
π_p	total potential energy
ρ	mass density of a material
ρ_w	weight density of a material
ω	angular velocity and natural circular frequency
Ω	potential energy of forces
ϕ	fluid head or potential, or rotation or slope in a beam
σ	normal stress
$\{\sigma_T\}$	thermal stress matrix
τ	shear stress and period of vibration
θ	angle between the x axis and the local x' axis for two-dimensional problems
θ_p	principal angle
$\theta_x, \theta_y, \theta_z$	angles between the global x, y, and z axes and the local x' axis, respectively, or rotations about the x and y axes in a plate
$[\psi]$	general displacement function matrix

Other Symbols

$\frac{d(\ )}{dx}$	derivative of a variable with respect to x
dt	time differential
$(\dot{\ })$	the dot over a variable denotes that the variable is being differentiated with respect to time
$[\ ]$	denotes a rectangular or a square matrix
$\{\ \}$	denotes a column matrix
$(\underline{\ })$	the underline of a variable denotes a matrix
$(\ ')$	the prime next to a variable denotes that the variable is being described in a local coordinate system
$[\ ]^{-1}$	denotes the inverse of a matrix
$[\ ]^T$	denotes the transpose of a matrix
$\frac{\partial(\ )}{\partial x}$	partial derivative with respect to x
$\frac{\partial(\ )}{\partial\{x\}}$	partial derivative with respect to each variable in $\{d\}$

CHAPTER 1

Introduction

CHAPTER OBJECTIVES

- To present an introduction to the finite element method.
- To provide a brief history of the finite element method.
- To introduce matrix notation.
- To describe the role of the computer in the development of the finite element method.
- To present the general steps used in the finite element method.
- To illustrate the various types of elements used in the finite element method.
- To show typical applications of the finite element method.
- To summarize some of the advantages of the finite element method.

Prologue

The finite element method is a numerical method for solving problems of engineering and mathematical physics. Typical problem areas of interest in engineering and mathematical physics that are solvable by use of the finite element method include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential.

For physical systems involving complicated geometries, loadings, and material properties, it is generally not possible to obtain analytical mathematical solutions to simulate the response of the physical system. Analytical solutions are those given by a mathematical expression that yields the values of the desired unknown quantities at any location in a body (here total structure or physical system of interest) and are thus valid for an infinite number of locations in the body. These analytical solutions generally require the solution of ordinary or partial differential equations, typically created by engineers, physicists, and mathematicians to eliminate the need for the creation and testing of numerous prototype designs, which may be quite costly. Because of the complicated geometries, loadings, and material properties, the solution to these differential equations is usually not obtainable. Hence, we need to rely on numerical methods, such as the finite element method, that can approximate the solution to these equations.

The finite element formulation of the problem results in a system of simultaneous algebraic equations for solution, rather than requiring the solution of differential equations. These numerical

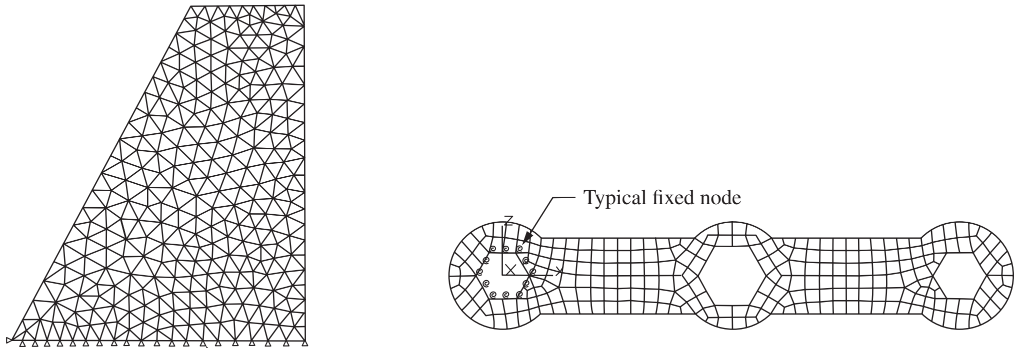


Figure 1–1 Two-dimensional models of (a) discretized dam and (b) discretized bicycle wrench (Applied loads are not shown.) All elements and nodes lie in a plane.

methods yield approximate values of the unknowns at discrete numbers of points in the continuum. Hence, this process of modeling a body by dividing it into an equivalent system of smaller bodies of units (finite elements) interconnected at points common to two or more elements (nodal points or nodes) and/or boundary lines and/or surfaces is called *discretization*. Figure 1–1 shows a cross section of a concrete dam and a bicycle wrench, respectively, that illustrate this process of discretization, where the dam has been divided into 490 plane triangular elements and the wrench has been divided into 254 plane quadrilateral elements. In both models the elements are connected at nodes and along inter element boundary lines. In the finite element method, instead of solving the problem for the entire body in one operation, we formulate the equations for each finite element and then combine them to obtain the solution for the whole body.

Briefly, the solution for structural problems typically refers to determining the displacements at each node and the stresses within each element making up the structure that is subjected to applied loads. In nonstructural problems, the nodal unknowns may, for instance, be temperatures or fluid pressures due to thermal or fluid fluxes.

This chapter first presents a brief history of the development of the finite element method. You will see from this historical account that the method has become a practical one for solving engineering problems only in the past 60 years (paralleling the developments associated with the modern high-speed electronic digital computer). This historical account is followed by an introduction to matrix notation; then we describe the need for matrix methods (as made practical by the development of the modern digital computer) in formulating the equations for solution. This section discusses both the role of the digital computer in solving the large systems of simultaneous algebraic equations associated with complex problems and the development of numerous computer programs based on the finite element method. Next, a general description of the steps involved in obtaining a solution to a problem is provided. This description includes discussion of the types of elements available for a finite element method solution. Various representative applications are then presented to illustrate the capacity of the method to solve problems, such as those involving complicated geometries, several different materials, and irregular loadings. Chapter 1 also lists some of the advantages of the finite element method in solving problems of engineering and mathematical physics. Finally, we present numerous features of computer programs based on the finite element method.

1.1 Brief History

This section presents a brief history of the finite element method as applied to both structural and nonstructural areas of engineering and to mathematical physics. References cited here are intended to augment this short introduction to the historical background.

The modern development of the finite element method began in the 1940s in the field of structural engineering with the work by Hrennikoff [1] in 1941 and McHenry [2] in 1943, who used a lattice of line (one-dimensional) elements (bars and beams) for the solution of stresses in continuous solids. In a paper published in 1943 but not widely recognized for many years, Courant [3] proposed setting up the solution of stresses in a variational form. Then he introduced piecewise interpolation (or shape) functions over triangular subregions making up the whole region as a method to obtain approximate numerical solutions. In 1947 Levy [4] developed the flexibility or force method, and in 1953 his work [5] suggested that another method (the stiffness or displacement method) could be a promising alternative for use in analyzing statically redundant aircraft structures. However, his equations were cumbersome to solve by hand, and thus the method became popular only with the advent of the high-speed digital computer.

In 1954 Argyris and Kelsey [6, 7] developed matrix structural analysis methods using energy principles. This development illustrated the important role that energy principles would play in the finite element method.

The first treatment of two-dimensional elements was by Turner et al. [8] in 1956. They derived stiffness matrices for truss elements, beam elements, and two-dimensional triangular and rectangular elements in plane stress and outlined the procedure commonly known as the *direct stiffness method* for obtaining the total structure stiffness matrix. Along with the development of the high-speed digital computer in the early 1950s, the work of Turner et al. [8] prompted further development of finite element stiffness equations expressed in matrix notation. The phrase *finite element* was introduced by Clough [9] in 1960 when both triangular and rectangular elements were used for plane stress analysis.

A flat, rectangular-plate bending-element stiffness matrix was developed by Melosh [10] in 1961. This was followed by development of the curved-shell bending-element stiffness matrix for axisymmetric shells and pressure vessels by Grafton and Strome [11] in 1963.

Extension of the finite element method to three-dimensional problems with the development of a tetrahedral stiffness matrix was done by Martin [12] in 1961, by Gallagher et al. [13] in 1962, and by Melosh [14] in 1963. Additional three-dimensional elements were studied by Argyris [15] in 1964. The special case of axisymmetric solids was considered by Clough and Rashid [16] and Wilson [17] in 1965.

Most of the finite element work up to the early 1960s dealt with small strains and small displacements, elastic material behavior, and static loadings. However, large deflection and thermal analysis were considered by Turner et al. [18] in 1960 and material nonlinearities by Gallagher et al. [13] in 1962, whereas buckling problems were initially treated by Gallagher and Padlog [19] in 1963. Zienkiewicz et al. [20] extended the method to visco elasticity problems in 1968.

In 1965 Archer [21] considered dynamic analysis in the development of the consistent-mass matrix, which is applicable to analysis of distributed-mass systems such as bars and beams in structural analysis.

With Melosh's [14] realization in 1963 that the finite element method could be set up in terms of a variational formulation, it began to be used to solve nonstructural applications. Field problems, such as determination of the torsion of a shaft, fluid flow, and heat conduction, were solved by Zienkiewicz and Cheung [22] in 1965, Martin [23] in 1968, and Wilson and Nickel [24] in 1966.

Further extension of the method was made possible by the adaptation of weighted residual methods, first by Szabo and Lee [25] in 1969 to derive the previously known elasticity equations used in structural analysis and then by Zienkiewicz and Parekh [26] in 1970 for transient field problems. It was then recognized that when direct formulations and variational formulations are difficult or not possible to use, the method of weighted residuals may at times be appropriate. For example, in 1977 Lyness et al. [27] applied the method of weighted residuals to the determination of magnetic field.

In 1976, Belytschko [28, 29] considered problems associated with large-displacement nonlinear dynamic behavior and improved numerical techniques for solving the resulting systems of equations. For more on these topics, consult the texts by Belytschko, Liu, Moran [58], and Crisfield [61, 62].

A relatively new field of application of the finite element method is that of bioengineering [30, 31]. This field is still troubled by such difficulties as nonlinear materials, geometric nonlinearities, and other complexities still being discovered.

From the early 1950s to the present, enormous advances have been made in the application of the finite element method to solve complicated engineering problems. Engineers, applied mathematicians, and other scientists will undoubtedly continue to develop new applications. For an extensive bibliography on the finite element method, consult the work of Kardestuncer [32], Clough [33], or Noor [57].

1.2 Introduction to Matrix Notation

Matrix methods are a necessary tool used in the finite element method for purposes of simplifying the formulation of the element stiffness equations, for purposes of longhand solutions of various problems, and, most important, for use in programming the methods for high-speed electronic digital computers. Hence matrix notation represents a simple and easy-to-use notation for writing and solving sets of simultaneous algebraic equations.

Appendix A discusses the significant matrix concepts used throughout the text. We will present here only a brief summary of the notation used in this text.

A **matrix** *is a rectangular array of quantities arranged in rows and columns that is often used as an aid in expressing and solving a system of algebraic equations*. As examples of matrices that will be described in subsequent chapters, the force components $(F_{1x}, F_{1y}, F_{1z}, F_{2x}, F_{2y}, F_{2z}, \ldots, F_{nx}, F_{ny}, F_{nz})$ acting at the various nodes or points $(1, 2, \ldots, n)$ on a structure and the corresponding set of nodal displacements $(u_1, v_1, w_1, u_2, v_2, w_2, \ldots, u_n, v_n, w_n)$ can both be expressed as matrices:

$$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ F_{2x} \\ F_{2y} \\ F_{2z} \\ \vdots \\ F_{nx} \\ F_{ny} \\ F_{nz} \end{Bmatrix} \qquad \{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix} \tag{1.2.1}$$

The subscripts to the right of F identify the node and the direction of force, respectively. For instance, F_{1x} denotes the force at node 1 applied in the x direction. The x, y, and z displacements at a node are denoted by u, v, and w, respectively. The subscript next to u, v, and w denotes the node. For instance, u_1, v_1, and w_1 denote the displacement components in the x, y, and z directions, respectively, at node 1. The matrices in Eqs. (1.2.1) are called *column matrices* and have a size of $n \times 1$. The brace notation { } will be used throughout the text to denote a column matrix. The whole set of force or displacement values in the column matrix is simply represented by $\{F\}$ or $\{d\}$.

The more general case of a known rectangular matrix will be indicated by use of the bracket notation [ ]. For instance, the element and global structure stiffness matrices $[k]$ and $[K]$, respectively, developed throughout the text for various element types (such as those in Figure 1–2 on page 11), are represented by square matrices given as

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \tag{1.2.2}$$

and

$$[K] = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \tag{1.2.3}$$

where, in structural theory, the elements k_{ij} and K_{ij} are often referred to as *stiffness influence coefficients*.

You will learn that the global nodal forces $\{F\}$ and the global nodal displacements $\{d\}$ are related through use of the global stiffness matrix $[K]$ by

$$\{F\} = [K]\{d\} \tag{1.2.4}$$

Equation (1.2.4) is called the *global stiffness equation* and represents a set of simultaneous equations. It is the basic equation formulated in the stiffness or displacement method of analysis.

To obtain a clearer understanding of elements K_{ij} in Eq. (1.2.3), we use Eq. (1.2.1) and write out the expanded form of Eq. (1.2.4) as

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ \vdots \\ F_{nz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & & & \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ w_n \end{Bmatrix} \tag{1.2.5}$$

Now assume a structure to be forced into a displaced configuration defined by $u_1 = 1, v_1 = w_1 = \cdots w_n = 0$. Then from Eq. (1.2.5), we have

$$F_{1x} = K_{11} \qquad F_{1y} = K_{21}, \ldots, F_{nz} = K_{n1} \tag{1.2.6}$$

Equations (1.2.6) contain all elements in the first column of $[K]$. In addition, they show that these elements, $K_{11}, K_{21}, \ldots, K_{n1}$, are the values of the full set of nodal forces required to maintain the imposed displacement state. In a similar manner, the second column in $[K]$ represents the values of forces required to maintain the displaced state $v_1 = 1$ and all other nodal displacement components equal to zero. We should now have a better understanding of the meaning of stiffness influence coefficients.

Subsequent chapters will discuss the element stiffness matrices $[k]$ for various element types, such as bars, beams, plane stress, and three-dimensional stress. They will also cover the procedure for obtaining the global stiffness matrices $[K]$ for various structures and for solving Eq. (1.2.4) for the unknown displacements in matrix $\{d\}$.

Using matrix concepts and operations will become routine with practice; they will be valuable tools for solving small problems longhand. And matrix methods are crucial to the use of the digital computers necessary for solving complicated problems with their associated large number of simultaneous equations.

1.3 Role of the Computer

As we have said, until the early 1950s, matrix methods and the associated finite element method were not readily adaptable for solving complicated problems. Even though the finite element method was being used to describe complicated structures, the resulting large number of algebraic equations associated with the finite element method of structural analysis made the method extremely difficult and impractical to use. However, with the advent of the computer, the solution of thousands of equations in a matter of minutes became possible.

The first modern-day commercial computer appears to have been the UNIVAC, IBM 701, which was developed in the 1950s. This computer was built based on vacuum-tube technology.

Along with the UNIVAC came the punch-card technology whereby programs and data were created on punch cards. In the 1960s, transistor-based technology replaced the vacuum-tube technology due to the transistor's reduced cost, weight, and power consumption and its higher reliability. From 1969 to the late 1970s, integrated circuit-based technology was being developed, which greatly enhanced the processing speed of computers, thus making it possible to solve larger finite element problems with increased degrees of freedom. From the late 1970s into the 1980s, large-scale integration as well as workstations that introduced a windows-type graphical interface appeared along with the computer mouse. The first computer mouse received a patent on November 17, 1970. Personal computers had now become mass-market desktop computers. These developments came during the age of networked computing, which brought the Internet and the World Wide Web. In the 1990s the Windows operating system was released, making IBM and IBM-compatible PCs more user friendly by integrating a graphical user interface into the software.

The development of the computer resulted in the writing of computational programs. Numerous special-purpose and general-purpose programs have been written to handle various complicated structural (and nonstructural) problems. Programs such as [46–56] illustrate the elegance of the finite element method and reinforce understanding of it.

In fact, finite element computer programs now can be solved on single-processor machines, such as a single desktop or laptop personal computer (PC) or on a cluster of computer nodes. The powerful memories of the PC and the advances in solver programs have made it possible to solve problems with over a million unknowns.

To use the computer, the analyst, having defined the finite element model, inputs the information into the computer. This information may include the position of the element nodal coordinates, the manner in which elements are connected, the material properties of the elements, the applied loads, boundary conditions, or constraints, and the kind of analysis to be performed. The computer then uses this information to generate and solve the equations necessary to carry out the analysis.

1.4 General Steps of the Finite Element Method

This section presents the general steps included in a finite element method formulation and solution to an engineering problem. We will use these steps as our guide in developing solutions for structural and nonstructural problems in subsequent chapters.

For simplicity's sake, for the presentation of the steps to follow, we will consider only the structural problem. The nonstructural heat-transfer, fluid mechanics, and electrostatics problems and their analogies to the structural problem are considered in Chapters 13 and 14.

Typically, for the structural stress-analysis problem, the engineer seeks to determine displacements and stresses throughout the structure, which is in equilibrium and is subjected to applied loads. For many structures, it is difficult to determine the distribution of deformation using conventional methods, and thus the finite element method is necessarily used.

There are three primary methods that can be used to derive the finite element equations of a physical system. These are (1) the *direct method* or *direct equilibrium method for structural analysis problems*, (2) the *variational methods consisting of among the subsets energy methods and the principle of virtual work*, and (3) the *weighted residual methods*. We briefly describe these three primary methods as follows, and more details of each will be described later in this section under step 4.

Direct Methods

The direct method, being the simplest and yielding a clear physical insight into the finite element method, is recommended in the initial stages of learning the concepts of the finite element method. However, the direct method is limited in its application to deriving element stiffness matrices for one-dimensional elements involving springs, uniaxial bars, trusses, and beams.

There are two general direct approaches traditionally associated with the finite element method as applied to structural mechanics problems. One approach, called the *force*, or *flexibility*, *method*, uses internal forces as the unknowns of the problem. To obtain the governing equations, first the equilibrium equations are used. Then necessary additional equations are found by introducing compatibility equations. The result is a set of algebraic equations for determining the redundant or unknown forces.

The second approach, called the *displacement*, or *stiffness*, *method*, assumes the displacements of the nodes as the unknowns of the problem. For instance, compatibility conditions requiring that elements connected at a common node, along a common edge, or on a common surface before loading remain connected at that node, edge, or surface after deformation takes place are initially satisfied. Then the governing equations are expressed in terms of nodal displacements using the equations of equilibrium and an applicable law relating forces to displacements.

These two direct approaches result in different unknowns (forces or displacements) in the analysis and different matrices associated with their formulations (flexibilities or stiffnesses). It has been shown [34] that, for computational purposes, the displacement (or stiffness) method is more desirable because its formulation is simpler for most structural analysis problems. Furthermore, a vast majority of general-purpose finite element programs have incorporated the displacement formulation for solving structural problems. Consequently, only the displacement method will be used throughout this text.

Variational Methods

The variational method is much easier to use for deriving the finite element equations for two- and three-dimensional elements than the direct method. However, it requires the existence of a functional, that upon minimizing yields the stiffness matrix and related element equations. For structural/stress analysis problems, we can use the principle of minimum potential energy as the functional, for this principle is a relatively easy physical concept to understand and has likely been introduced to the reader in an undergraduate course in basic applied mechanics [35].

It can be used to develop the governing equations for both structural and nonstructural problems. The variational method includes a number of principles. One of these principles, used extensively throughout this text because it is relatively easy to comprehend and is often introduced in basic mechanics courses, is the theorem of minimum potential energy that applies to materials behaving in a linear-elastic manner. This theorem is explained and used in various sections of the text, such as Section 2.6 for the spring element, Section 3.10 for the bar element, Section 4.7 for the beam element, Section 6.2 for the constant strain triangle plane stress and plane strain element, Section 9.1 for the axisymmetric element, Section 11.2 for the three-dimensional solid tetrahedral element, and Section 12.2 for the plate bending element. A functional analogous to that used in the theorem of minimum potential energy is then employed to develop the finite element equations for the nonstructural problem of heat transfer presented in Chapter 13.

Another variational principle often used to derive the governing equations is the principle of virtual work. This principle applies more generally to materials that behave in a linear-elastic fashion, as well as those that behave in a nonlinear fashion. The principle of virtual work is described in Appendix E for those choosing to use it for developing the general governing finite element equations that can be applied specifically to bars, beams, and two- and three-dimensional solids in either static or dynamic systems.

Weighted Residual Methods

The weighted residual methods [36] allow the finite element method to be applied directly to any differential equation without having the existence of a variational principle. Section 3.12 introduces the Galerkin method (a very well-known residual method) for deriving the bar element stiffness matrix and associated element equations. Section 3.13 introduces other residual methods for solving the governing differential equation for the axial displacement along a bar.

The finite element method involves modeling the structure using small interconnected elements called *finite elements*. A displacement function is associated with each finite element. Every interconnected element is linked, directly or indirectly, to every other element through common (or shared) interfaces, including nodes and/or boundary lines and/or surfaces. By using known stress/strain properties for the material making up the structure, one can determine the behavior of a given node in terms of the properties of every other element in the structure. The total set of equations describing the behavior of each node results in a series of algebraic equations best expressed in matrix notation.

We now present the steps, along with explanations necessary at this time, used in the finite element method formulation and solution of a structural problem. The purpose of setting forth these general steps now is to expose you to the procedure generally followed in a finite element formulation of a problem. You will easily understand these steps when we illustrate them specifically for springs, bars, trusses, beams, plane frames, plane stress, axisymmetric stress, three-dimensional stress, plate bending, heat transfer, fluid flow, and electrostatics in subsequent chapters. We suggest that you review this section periodically as we develop the specific element equations.

Keep in mind that the analyst must make decisions regarding dividing the structure or continuum into finite elements and selecting the element type or types to be used in the analysis (step 1), the kinds of loads to be applied, and the types of boundary conditions or supports to be applied. The other steps, 2 through 7, are carried out automatically by a computer program.

Step 1 Discretize and Select the Element Types

Step 1 involves dividing the body into an equivalent system of finite elements with associated nodes and choosing the most appropriate element type to model most closely the actual physical behavior. The total number of elements used and their variation in size and type within a given body are primarily matters of engineering judgment. The elements must be made small enough to give usable results and yet large enough to reduce computational effort. Small elements (and possibly higher-order elements) are generally desirable where the results are changing rapidly, such as where changes in geometry occur; large elements can be used where results are relatively constant. We will have more to say about discretization guidelines in later chapters, particularly in Chapter 7, where the concept becomes quite significant.