6TH EDITION

MATHEMATICS EXPLAINED FOR PRIMARY TEACHERS

DEREK HAYLOCK with RALPH MANNING

MATHEMATICS EXPLAINED FOR PRIMARY TEACHERS

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PRAISE FOR THE FIFTH EDITION

'Some students refer to the book as their "maths bible" and I have spoken to and have had e-mails from former students who continue to refer to it (we recommend that they buy it from Day One), saying "Thank goodness for Haylock!"'

Debra Crouch, Senior Lecturer in Primary Mathematics, University of Middlesex

'As always this book will prove to be an invaluable asset not only to trainee teachers, but as many have agreed also, after school experience placements. There is a growing awareness in primary schools of this book being an asset for all members of staff no matter whether they have two or twenty-two years of classroom experience.'

Andrew Gallacher, Mathematics Co-ordinator for Secondary PGDE, University of Glasgow'

'The physical structure of the chapters are well presented, and there is a good balance of colour, text, symbols, and pictorial representations. Access to the content is well supported through the chapter overview that provides a ready reference point for locating the content. Within the chapters, each element is easily located through the use of clear headings. Explicit examples are used to clarify the meaning of the mathematical terminology and the structure of the Learning and Teaching Point provides a clear summary of the elements in the corresponding text. Good visual supports, such as number lines, base-ten blocks and arrow cards, supplement the written text and consolidate understanding. Throughout the text, common misconceptions are identified and comprehensively explained. The glossary of terminology at the end of each chapter is a useful and well used resource.'

Theresa Shellshear, Mathematics Lecturer, Australian Catholic University

'This book can rightly be considered the ultimate reference for securing appropriate mathematics subject knowledge for primary teaching. In the chapter 'Written Methods for Multiplication and Division' the concepts behind the methods are explained clearly and thoroughly. Of particular use are the 'teaching points' sections and links to video explanations and further practice. It is clear that this book is written *by* a teacher for teachers and its main objective is to ensure teachers understand the concepts behind what that they teach so that children will not just *learn* mathematics but *understand* mathematics. This book will be a key text for primary BA Ed and PGCE students.'

Rachel Wallis, Lead in Primary Mathematics and Numeracy, University of Wales Trinity Saint David

ABOUT THE AUTHORS

DEREK HAYLOCK worked for over 30 years in teacher education, both initial and in-service. He was Co-Director of Primary Initial Teacher Training and responsible for the mathematics components of the primary programmes at the University of East Anglia (UEA), Norwich. He is currently a Senior Fellow in Education at UEA. He has taught mathematics at every level and, in particular, has considerable practical experience of teaching and researching in primary classrooms. His commitment to mathematics being learnt with understanding, relevance and enjoyment has underpinned his work as a writer, consultant and professional speaker. As well as his extensive publications in the field of education, he has written seven books of Christian drama for young people and written and composed a Christmas musical (all published by Church House/National Society).

RALPH MANNING has contributed the online activities for lesson plans that are linked to each of Chapters 6 to 28. He left a former career in Information Technology to become a primary teacher, before lecturing in teacher education at UEA, Norwich. He now works as an education consultant.

OTHER BOOKS IN EDUCATION BY DEREK HAYLOCK

Browne, A. and Haylock, D. (eds) (2004) *Professional Issues for Primary Teachers*. London: Sage Publications.

Haylock, D. (1991) *Teaching Mathematics to Low Attainers 8–12*. London: Sage Publications.

Haylock, D. (2001) *Numeracy for Teaching*. London: Sage Publications.

Haylock, D. (2019) *Student Workbook for Mathematics Explained for Primary Teachers*, 6th edn. London: Sage Publications.

Haylock, D. and Cockburn, A. (2017) *Understanding Mathematics for Young Children*, 5th edn. London: Sage Publications.

Haylock, D. and D'Eon, M. (1999) *Helping Low Achievers Succeed at Mathematics*. Toronto: Trifolium Books.

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Haylock, D. with Thangata, F. (2007) *Key Concepts in Teaching Primary Mathematics*. London: Sage Publications.

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‘HOW TO’ GUIDE

‘Hi! We are the SAGE Primary Mathematics Student Panel. We have road tested the chapter features and online resources for this book to make sure they support you with your studies.

This book is brilliant – it is easy to understand and provides helpful hints and tips. It is a really convenient way to support your mathematical understanding. You can find these helpful resources when you download your free interactive ebook – which is so quick and easy to do!

Here’s how these easily accessible resources can help you:’

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These links are easily accessible and the interactive format breaks down the curriculum requirements. This makes lesson planning and assignment writing easier as you have all your info in one easy-to-access place.
- **Amber**

SECTION INTRODUCTION VIDEOS

You can explore an interactive introduction, which gives you an overview on what the chapter is about and acts as a good starting point to develop or refresh your knowledge.
- **Andre**

PROBLEM SOLVED! VIDEOS

These animated videos show you how to work through different maths methods to close gaps in your understanding.
- **Robert and Olivia**

SELF-ASSESSMENT QUESTIONS

These enable you to revise the knowledge you have learnt, consolidate your learning and determine whether you have understood the concepts and can move on, or whether you need to revise.
- **Andre**

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These save time – great for lesson planning! Gives you lesson ideas and ways of approaching concepts in a classroom context.
- **Sarah & Olivia**

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Test where your knowledge currently sits and use these for additional study once the chapter has finished. Practice makes perfect!
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INTRODUCTION

This sixth edition has been revised to ensure that the content is in line with the mathematics programmes of study for Key Stages 1 and 2 (children aged 5 to 11 years) in England (DfE, 2013) and consistent with the current emphasis on teaching for mastery, promoted by the National Council for Excellence in Teaching Mathematics. As well as ensuring that I address all the statutory requirements for mathematics in primary schools in England that have to teach the National Curriculum, I am aware that many readers of this book will teach mathematics to this age range of children in other countries with a variety of statutory curriculums and also that there are many schools in England that are not required to teach the English National Curriculum. It has been encouraging to hear that teachers, teacher trainees and mathematics educators in many other countries have found the previous editions to be helpful and relevant to their work. I am confident that this will continue to be the case. In particular, I have ensured that the primary national curriculums of the other countries in the United Kingdom are covered comprehensively.

I have continued in my commitment to focus on what has always been the key message of Mathematics Explained: the need for priority to be given in initial teacher training and professional development to primary school teachers developing secure and comprehensive subject knowledge in mathematics, characterized by understanding and awareness of the pedagogical implications. Even well-qualified graduates feel insecure and uncertain about much of the mathematics they have to teach, as is demonstrated in Chapter 1 of the book. I know from experience just how much they appreciate a systematic explanation of even the most elementary mathematical concepts and procedures of the primary curriculum. In my long career in teacher education, I have often reflected on what qualities make a good teacher. I have a little list. Top of the list is the following conviction: The best teachers have a secure personal understanding of the structure and principles of what they are teaching. This book is written to help primary teachers, present and future, to achieve this in mathematics. It sets out to explain the subject to primary school teachers, so that they in turn will have the confidence to provide appropriate, systematic and careful explanation of mathematical ideas and procedures to their pupils, with an emphasis on the development of understanding, rather than mere learning by rote. This is always done from the perspective of how children learn, understand and develop mastery of this subject. Implications for learning and teaching are embedded in the text and highlighted as 'Learning and teaching points' distributed throughout each chapter.

Section A (Chapters 1–3) of this book is about Mathematical Understanding. Chapter 1, drawing on my research with trainee teachers, provides evidence for the need to develop understanding of mathematics and to lower anxiety about this subject in those who are to teach in primary schools. Chapter 2 considers the distinctive contribution that mathematics makes to the primary curriculum; and Chapter 3 – which I consider to be the key chapter in this book – is about pupils learning to learn mathematics with understanding.

Section B explains the key features of mathematical reasoning and problem solving and seeks to give these two major themes in the mathematics curriculum the prominence and priority they warrant.

Sections C–H then focus on the content and principles of various sections of primary school mathematics, covering all you need to understand about: numbers, including the number system and various number properties; calculations, mental and written, including those with fractions, decimals and percentages; algebra; the principles and concepts of measurement and units of measurement; geometry; and statistics and probability.

It is important that those who teach mathematics to children know and understand more mathematics than the children may have to learn. This is necessary so that the teacher can feel confident and teach with authority in this area of the curriculum, but also so that they are more aware of the significance of what the children are learning. So, in places in the book there will be material that may go beyond the immediate requirements of the primary school curriculum. In particular, some sections – such as some calculations with decimals and percentages and much of the material in Chapter 27 – are provided for the personal development of the primary teacher, to help them to handle, with confidence, some of the encounters with mathematics they will have in their professional role.

Finally: a comment about calculators. Although calculators have effectively been outlawed from primary schools that are constrained by the National Curriculum in England, they are still available in many other primary schools in this country as well as in classes elsewhere in the UK and in countries overseas. I have therefore included here and there a number of ways in which simple four-function calculators can be used effectively to provide insights and to promote understanding in learning mathematics. All readers should find that engagement with these examples proves to enhance their own understanding of the mathematical ideas being explained.

Derek Haylock, Norwich

SECTION A

MATHEMATICAL UNDERSTANDING

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1

PRIMARY TEACHERS' INSECURITY ABOUT MATHEMATICS

IN THIS CHAPTER, THERE ARE EXPLANATIONS OF

- the importance of primary school teachers really understanding the mathematics they teach and being able to explain it clearly to the children they teach;
- the relationship of mathematics anxiety to avoidance of mathematical demand, rote learning and low levels of creativity in problem solving;
- attitudes of adults in general toward mathematics;
- mathematics anxiety in primary school teachers;
- the insecurity about mathematics of many primary trainee teachers.

UNDERSTANDING AND EXPLAINING

Being a successful learner in mathematics involves constructing *understanding* through exploration, problem solving, discussion and practical experience – and also through interaction with a teacher who has a clear grasp of the underlying structure of the mathematics being learnt. For children to enjoy learning mathematics, it is essential that they should understand it; and that they should make sense of what they are doing in the subject, and not just learn to reproduce learnt procedures and recipes that are low in meaningfulness and purposefulness.

One of the ways for children to learn and understand much of the mathematics in the primary school curriculum is for a teacher who understands it to explain it to them. Those who teach mathematics in primary schools should ensure that the approach they take to organizing children's activities allows sufficient opportunities for them to provide teaching that includes engaging with children in question and answer, discussion and explanation – all aimed at promoting understanding and confidence in mathematics. Of course, there is more to learning mathematics than just a teacher explaining something and then following this up with exercises. The key processes of mathematical reasoning, applying mathematics and problem solving must always be at the heart of learning the subject – and these figure prominently in this book, particularly in Section B (Chapters 4 and 5).

But children do need 'explanation' to help them to understand mathematics, to make sense of it, to give them confidence and to help them have positive attitudes to the subject. There is now in England a greater awareness that primary teachers must organize their lessons and the children's activities in ways that give opportunities for them to provide careful, systematic and appropriate explanation of mathematical concepts, procedures and principles to groups of children. That many primary teachers have in the past neglected this aspect of teaching may possibly have been associated with a prevailing primary ethos which perhaps over-emphasized active learning and children discovering things for themselves. But it seems often to me to be a consequence of the teacher's own insecurity and anxiety about mathematics, which are characteristics of too many primary school teachers. This book is written to equip teachers with the knowledge, understanding and confidence they require to be able to explain mathematical ideas to the children they teach. English primary schools, following the example of some higher-performing countries in terms of mathematics achievement, are now putting an emphasis on teaching for what is termed 'mastery' in mathematics (see Chapter 3). Mastery approaches in teaching mathematics aim for a deep understanding of mathematical procedures, concepts and principles for all children. It is self-evident that a prerequisite for teaching for mastery is that primary school teachers themselves have this deep and secure understanding of the mathematics in the primary school curriculum. Helping teachers and teacher trainees to feel confident in their own understanding and mastery of this subject is the principal purpose of this book.

ATTITUDES TO MATHEMATICS IN ADULTS

There are widespread confusions amongst the adult population in Britain about many of the basic mathematical processes of everyday life. This lack of confidence in basic mathematics appears to be related to the anxiety about mathematics and feelings of inadequacy in this subject that are common amongst the adult population. These phenomena have been demonstrated by surveys of adults' attitudes to mathematics (for example, Coben et al., 2003). Findings indicate that many adults, in relation to mathematical tasks, admit to feelings of anxiety, helplessness, fear, dislike and even guilt. The feeling of guilt is particularly marked amongst those with high academic qualifications, who feel that they ought to be more confident in their understanding of this subject. There is a perception that there are proper ways of doing mathematics and that the subject is characterized by questions to which your answers are either right or wrong. Feelings of failure, frustration and anxiety are identified by many adults as having their roots in unsympathetic attitudes of teachers and the expectations of parents. A project at King's College, London, looking at the attitudes of adults attending numeracy classes, found that the majority of such adults viewed themselves as failures and carried various types of emotional baggage from their school-days. They spoke of their poor experience of schooling and of feeling that they had been written off by their mathematics teachers, usually at an early stage. Their return to the mathematics classroom as adults was accompanied by feelings of anxiety, even fear (Swain, 2004). Significantly, in a survey of over 500 adults in the UK, Lim (2002) identified three widely claimed myths about mathematics: it is a difficult subject; it is only for clever people; and it is a male domain.

MATHEMATICS ANXIETY IN TEACHERS AND LEARNERS

Research into primary school teachers' attitudes to mathematics reveals that many primary teachers experience the same kinds of feelings of panic and anxiety when faced with unfamiliar mathematical tasks (Briggs, 1993), that they are muddled in their thinking about many of the basic mathematical concepts which underpin the material they teach to children, and that they are all too aware of their personal inadequacies in mathematics. The widespread view that mathematics is a difficult subject, and therefore only for clever people, increases these feelings of inadequacy – and the common perception that mathematics is a male domain exacerbates the problem within a subset of the teaching profession that continues to be largely populated by women. The importance of tackling these attitudes to the subject is underlined by the findings of Burnett and Wichman (1997) that primary teachers' (and parents') own anxieties about mathematics can often be passed on to the children they teach.

It is important not to generate mathematics anxiety in the children we teach, because high levels of anxiety affect a person's ability to perform to their potential. Boaler (2012) argues

that an unhelpful level of anxiety towards mathematics often starts early in the primary years of education and is found even amongst children who otherwise do well at school. She reports the well-established finding that a high level of mathematics anxiety leads to difficulties in learning the subject and avoidance of mathematical demand, and identifies in particular the way in which timed tests in mathematics contribute significantly to anxiety. The research of Ashcraft and Kirk (2001) and Ashcraft and Moore (2009), for example, confirms that raising anxiety about mathematics produces a drop in performance in the subject, particularly in terms of the individual's access to their 'working memory' and a tendency to avoid any kind of mathematical demand. Puteh (1998) has provided a helpful diagrammatic summary of research in this area (see Figure 1.1).

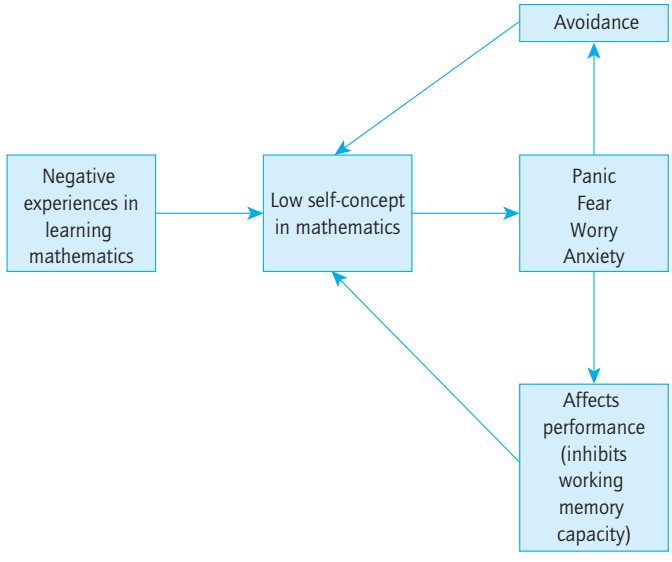

Figure 1.1 *Some factors associated with mathematics anxiety*

Source: Puteh (1998)

Newstead (1995, 1998) found that a teaching method that promoted understanding rather than just the memorization and rehearsal of procedures and recipes was associated significantly with lower levels of mathematics anxiety in primary school children. Other research, such as Ford et al. (2005), shows that high anxiety towards mathematics has a negative effect on learners' performance. In particular, anxiety leads to reliance on learning mathematics by rote, rather than aiming and expecting to learn with understanding; this is a vicious circle, because reliance on rote memorization of rules rather than understanding then increases anxiety when faced with anything unfamiliar (Figure 1.2).

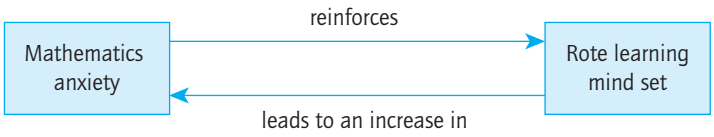

Figure 1.2 *Mathematics anxiety and rote learning*

In my own research I have found that anxiety about mathematics – which is reinforced when children are not being helped to learn mathematics in a meaningful way – is associated with rigid and inflexible thinking in unfamiliar mathematical tasks and leads to insecurity and caution when faced with a non-routine mathematical challenge, and therefore to low levels of creativity in problem solving (Figure 1.3). Creativity in mathematics is explained further in Chapter 4.

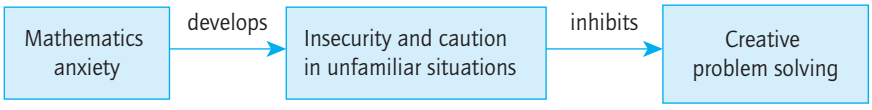

Figure 1.3 *Mathematics anxiety inhibits creativity in problem solving*

TRAINEE TEACHERS' ANXIETIES

The background for this book is mainly my experience of working with graduates enrolled on a one-year primary initial teacher-training programme. The trainee teachers I have worked with have been highly motivated, good honours graduates, with the subjects of their degree studies ranging across the curriculum. Over a number of years of working with such trainees, it became clear to me that many of them start their course with a high degree of anxiety about having to teach mathematics.

An invitation was given for any trainees who felt particularly worried about mathematics to join a group who would meet for an hour a week throughout their course, to discuss their anxieties and to identify which aspects of the National Curriculum for mathematics appropriate to the age range they would be teaching gave them most concern. A surprisingly large number of trainees turned up for these sessions. Discussions with them revealed both those aspects of mathematics anxiety that they still carry around with them, derived clearly from their own experiences of learning mathematics at school, and the anxiety related to specific areas of mathematics they will have to teach where they have doubts about their own understanding.

Below, I recount some of the statements made by the primary trainee teachers in my group about their attitudes towards and experiences of learning mathematics. In reading these comments, it is important to remember that these are students who have come through the system