Introduction to Operations Research

Frederick S. HILLIER

Gerald J. LIEBERMAN

Mc Graw Hill

ELEVENTH EDITION

INTRODUCTION TO OPERATIONS RESEARCH

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Eleventh Edition

FREDERICK S. HILLIER
Stanford University

GERALD J. LIEBERMAN
Late of Stanford University

Mc Graw Hill

INTRODUCTION TO OPERATIONS RESEARCH

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ABOUT THE AUTHORS

Frederick S. Hillier was born and raised in Aberdeen, Washington, where he was an award winner in statewide high school contests in essay writing, mathematics, debate, and music. As an undergraduate at Stanford University, he ranked first in his engineering class of over 300 students. He also won the McKinsey Prize for technical writing, won the Outstanding Sophomore Debater award, played in the Stanford Woodwind Quintet and Stanford Symphony Orchestra, and won the Hamilton Award for combining excellence in engineering with notable achievements in the humanities and social sciences. Upon his graduation with a BS degree in industrial engineering, he was awarded three national fellowships (National Science Foundation, Tau Beta Pi, and Danforth) for graduate study at Stanford with specialization in operations research. During his three years of graduate study, he took numerous additional courses in mathematics, statistics, and economics beyond what was required for his MS and PhD degrees while also teaching two courses (including "Introduction to Operations Research"). Upon receiving his PhD degree, he joined the faculty of Stanford University and began work on the 1st edition of this textbook two years later. He subsequently earned tenure at the age of 28 and the rank of full professor at 32. He also received visiting appointments at Cornell University, Carnegie-Mellon University, the Technical University of Denmark, the University of Canterbury (New Zealand), and the University of Cambridge (England). After 35 years on the Stanford faculty, he took early retirement from his faculty responsibilities in order to focus full time on textbook writing, and now is Professor Emeritus of Operations Research at Stanford.

Dr. Hillier's research has extended into a variety of areas, including integer programming, queueing theory and its application, statistical quality control, the application of operations research to the design of production systems, and capital budgeting. He has published widely, and his seminal papers have been selected for republication in books of selected readings at least 10 times. He was the first-prize winner of a research contest on "Capital Budgeting of Interrelated Projects" sponsored by The Institute of Management Sciences (TIMS) and the U.S. Office of Naval Research. He and Dr. Lieberman also received the honorable mention award for the 1995 Lanchester Prize (best English-language publication of any kind in the field of operations research), which was awarded by the Institute for Operations Research and the Management Sciences (INFORMS) for the 6th edition of this book. In addition, he was the recipient of the prestigious 2004 INFORMS Expository Writing Award for the 8th edition of this book.

Dr. Hillier has held many leadership positions with the professional societies in his field. For example, he has served as treasurer of the Operations Research Society of America (ORSA), vice president for meetings of TIMS, co-general chairman of the 1989 TIMS International Meeting in Osaka, Japan, chair of the TIMS Publications Committee, chair of the ORSA Search Committee for Editor of *Operations Research,* chair of the ORSA Resources Planning Committee, chair of the ORSA/TIMS Combined Meetings Committee, and chair of the John von Neumann Theory Prize Selection Committee for INFORMS. He also is a Fellow of INFORMS. In addition, he served for 20 years (until 2013) as the founding series editor for Springer's International Series in Operations Research and Management Science, a particularly prominent book series with nearly

300 published books. In 2018, he was awarded the Kimball Medal (a lifetime achievement award) by INFORMS for his distinguished contributions to the field and to INFORMS.

In addition to *Introduction to Operations Research* and two companion volumes, *Introduction to Mathematical Programming* (2nd ed., 1995) and *Introduction to Stochastic Models in Operations Research* (1990), his books are *The Evaluation of Risky Interrelated Investments* (North-Holland, 1969), *Queueing Tables and Graphs* (Elsevier North-Holland, 1981, co-authored by O. S. Yu, with D. M. Avis, L. D. Fossett, F. D. Lo, and M. I. Reiman), and *Introduction to Management Science: A Modeling and Case Studies Approach with Spreadsheets* (6th ed., McGraw-Hill, 2019, co-authored by his son Mark Hillier).

The late **Gerald J. Lieberman** sadly passed away in 1999. He had been Professor Emeritus of Operations Research and Statistics at Stanford University, where he was the founding chair of the Department of Operations Research. He was both an engineer (having received an undergraduate degree in mechanical engineering from Cooper Union) and an operations research statistician (with an AM from Columbia University in mathematical statistics, and a PhD from Stanford University in statistics).

Dr. Lieberman was one of Stanford's most eminent leaders. After chairing the Department of Operations Research, he served as associate dean of the School of Humanities and Sciences, vice provost and dean of research, vice provost and dean of graduate studies, chair of the faculty senate, member of the University Advisory Board, and chair of the Centennial Celebration Committee. He also served as provost or acting provost under three different Stanford presidents.

Throughout these years of university leadership, he also remained active professionally. His research was in the stochastic areas of operations research, often at the interface of applied probability and statistics. He published extensively in the areas of reliability and quality control, and in the modeling of complex systems, including their optimal design, when resources are limited.

Highly respected as a senior statesman of the field of operations research, Dr. Lieberman served in numerous leadership roles, including as the elected president of The Institute of Management Sciences. His professional honors included being elected to the National Academy of Engineering, receiving the Shewhart Medal of the American Society for Quality Control, receiving the Cuthbertson Award for exceptional service to Stanford University, and serving as a Fellow at Stanford's Center for Advanced Study in the Behavioral Sciences. In addition, the Institute for Operations Research and the Management Sciences (INFORMS) awarded him and Dr. Hillier the honorable mention award for the 1995 Lanchester Prize for the 6th edition of this book. In 1996, INFORMS also awarded him the prestigious Kimball Medal for his distinguished contributions to the field and to INFORMS.

In addition to *Introduction to Operations Research* and two companion volumes, *Introduction to Mathematical Programming* (2nd ed., 1995) and *Introduction to Stochastic Models in Operations Research* (1990), his books are *Handbook of Industrial Statistics* (Prentice-Hall, 1955, co-authored by A. H. Bowker), *Tables of the Non-Central t-Distribution* (Stanford University Press, 1957, co-authored by G. J. Resnikoff), *Tables of the Hypergeometric Probability Distribution* (Stanford University Press, 1961, co-authored by D. Owen), *Engineering Statistics*, (2nd ed., Prentice-Hall, 1972, co-authored by A. H. Bowker), and *Introduction to Management Science: A Modeling and Case Studies Approach with Spreadsheets* (McGraw-Hill/Irwin, 2000, co-authored by F. S. Hillier and M. S. Hillier).

ABOUT THE CASE WRITERS

Karl Schmedders is professor of quantitative business administration at the University of Zurich in Switzerland and a visiting associate professor at the Kellogg Graduate School of Management (Northwestern University). His research interests include management science, financial economics, and computational economics and finance. He received his PhD in operations research from Stanford University, where he taught both undergraduate and graduate classes in operations research, including a case studies course in operations research. He received several teaching awards at Stanford, including the university's prestigious Walter J. Gores Teaching Award. After post-doctoral research at the Hoover Institution, a think tank on the Stanford campus, he became assistant professor of managerial economics and decision sciences at the Kellogg School. He was promoted to associate professor in 2001 and received tenure in 2005. In 2008, he joined the University of Zurich, where he currently teaches courses in management science, business analytics, and computational economics and finance. He has published research articles in international academic journals such as *Management Science, Operations Research, Econometrics, The Review of Economic Studies,* and *The Journal of Finance*, among others. He is a co-founder of an EdTech Startup developing a digital learning and grading platform for science education. At Kellogg he received several teaching awards, including the L. G. Lavengood Professor of the Year Award. More recently he has won the best professor award of the Kellogg School's European EMBA program in eight different years, as well as in 2017 for its EMBA program in Hong Kong.

Molly Stephens is a partner in the Los Angeles office of Quinn, Emanuel, Urquhart & Sullivan, LLP. She graduated from Stanford University with a BS degree in industrial engineering and an MS degree in operations research. Ms. Stephens taught public speaking in Stanford's School of Engineering and served as a teaching assistant for a case studies course in operations research. As a teaching assistant, she analyzed operations research problems encountered in the real world and transformed these problems into classroom case studies. Her research was rewarded when she won an undergraduate research grant from Stanford to continue her work and was invited to speak at an INFORMS conference to present her conclusions regarding successful classroom case studies. Following graduation, Ms. Stephens worked at Andersen Consulting as a systems integrator, experiencing real cases from the inside, before resuming her graduate studies to earn a JD degree (with honors) from the University of Texas Law School at Austin. She is a partner in the largest law firm in the United States devoted solely to business litigation, where her practice focuses on complex financial and securities litigation. She also is ranked as a leading securities litigator by Chambers USA (2013 and 2014), which acknowledged "praise for her powerful and impressive securities litigation practice" and noted that she is "phenomenally bright, a critical thinker and great listener."

DEDICATION

To the memory of our parents

and

To the memory of my beloved mentor,
Gerald J. Lieberman, who was one of the true
giants of our field

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PREFACE

When Jerry Lieberman and I started working on the first edition of this book, our goal was to develop a pathbreaking textbook that would help establish the future direction of education in what was then the emerging field of operations research. Following publication, it was unclear how well this particular goal was met, but what did become clear was that the demand for the book was far larger than either of us had anticipated. Neither of us could have imagined that this extensive worldwide demand would continue at such a high level for such an extended period of time.

The enthusiastic response to our first ten editions has been most gratifying. It was a particular pleasure to have the field's leading professional society, the international Institute for Operations Research and the Management Sciences (INFORMS), award the 6th edition honorable mention for the 1995 INFORMS Lanchester Prize (the prize awarded for the year's most outstanding English-language publication of any kind in the field of operations research).

Then, just after the publication of the eighth edition, it was especially gratifying to be the recipient of the prestigious 2004 INFORMS Expository Writing Award for this book, including receiving the following citation:

> Over 37 years, successive editions of this book have introduced more than one-half million students to the field and have attracted many people to enter the field for academic activity and professional practice. Many leaders in the field and many current instructors first learned about the field via an edition of this book. The extensive use of international student editions and translations into 15 other languages has contributed to spreading the field around the world. The book remains preeminent even after 37 years. Although the eighth edition just appeared, the seventh edition had 46 percent of the market for books of its kind, and it ranked second in international sales among all McGraw-Hill publications in engineering.
>
> Two features account for this success. First, the editions have been outstanding from students' points of view due to excellent motivation, clear and intuitive explanations, good examples of professional practice, excellent organization of material, very useful supporting software, and appropriate but not excessive mathematics. Second, the editions have been attractive from instructors' points of view because they repeatedly infuse state-of-the-art material with remarkable lucidity and plain language. For example, a wonderful chapter on metaheuristics was created for the eighth edition.

When we began work on the first edition, Jerry already was a prominent member of the field, a successful textbook writer, and the chairman of a renowned operations research program at Stanford University. I was a very young assistant professor just starting my career. It was a wonderful opportunity for me to work with and to learn from the master. I will be forever indebted to Jerry for giving me this opportunity.

Now, sadly, Jerry is no longer with us. During the progressive illness that led to his death in 1999, I resolved that I would pick up the torch and devote myself to subsequent editions of this book, maintaining a standard that would fully honor Jerry. Therefore, I took early retirement from my faculty responsibilities at Stanford in order to work full time on textbook writing for the foreseeable future. This has enabled me to spend far more than the usual amount of time in preparing each new edition. It also has enabled

me to closely monitor new trends and developments in the field in order to bring this edition completely up to date. This monitoring has led to the addition of a considerable number of important topics to recent editions of the book.

The field continues to evolve fairly rapidly. The most important of the recent developments has been the rise of analytics as a very important complement to operations research. Other important trends also are under way. Therefore, I have made a special effort with this edition to continue bringing this book fully into the 21st century. The many major additions to the new edition are outlined below.

WHAT'S NEW IN THIS EDITION

- Added a New Section 1.3: The Relationship between Analytics and Operations Research.
- Added a New Section 1.5: Some Trends That Should Further Increase the Future Impact of Operations Research.
- Added a New Section 2.2: Gathering and Organizing Relevant Data.
- Added a New Section 2.3: Using Descriptive Analytics to Analyze Big Data.
- Added a New Section 2.4: Using Predictive Analytics to Analyze Big Data.
- Reorganized Section 4.6 (Adapting the Simplex Method to Nonstandard Forms) into Three New Shorter Sections.
- Section 4.10: Added Up-To-Date information on the Factors Affecting the Speed of the Simplex Method (and Its Variants).
- Section 4.11: Added Up-To-Date Information on the Factors Affecting the Relative Performance of the Simplex Method and Interior-Point Algorithms.
- Shortened and Revised Section 12.3: Using Binary Variables to Deal with Fixed Charges.
- Shortened and Revised Section 12.4: A Binary Representation of General Integer Variables.
- Added a New Section 16.7: Multiple Criteria Decision Analysis, Including Goal Programming.
- Added a New Section 17.11: Behavioral Queueing Theory.
- Added a New Section 19.4: Markov Decision Processes in Practice.
- Added a New Section 20.5: Simulation Optimization.
- Added many New Smaller Updates, Including New Application Vignettes and New Selected References.

Reductions to Make Room for All These New Additions:

The first edition of this book was only a little over 600 pages. However, subsequent editions kept growing until it reached 1200 pages with the 7th edition. That is much too large for an introductory textbook, so I have been working ever since to decrease the size of each new edition. I finally got the 10th edition down below 1000 pages again (excluding indices and front matter) and have made a real effort to reduce the size a little further with this new edition. This was a real challenge with all of the new additions outlined above. However, I feel that the reductions listed below have helped to make this a better book by enabling more focus on the important material.

- Dropped Analytic Solver Platform for Education. (This is an excellent software package, but Frontline Systems now is charging students to use it and reviewers expressed little interest in retaining it. This one reduction saved approximately 35 pages.)
- Eliminated an Overabundance of Linear Programming Formulation Examples in Section 3.4. (Dropping three of the six complicated examples saved 10 pages.)

- Shifted Section 6.2 (Economic Interpretation of Duality) to a Supplement on the Website.
- Shifted the General Procedure for Constructing an Initial BF Solution for the Transportation Simplex Method in Section 9.2 to a Supplement on the Website.
- Shifted Most of Section 12.3 (Innovative Uses of Binary Variables in Model Formulation) and Section 12.4 (Some Formulation Examples) to a Supplement on the Website, While Retaining More Elementary Material.
- Deleted a Subsection in Section 17.3 on Outdated Award-Winning Studies That Applied Queueing Theory.
- Deleted 14 Outdated Application Vignettes (While Also Adding 11 New Ones That Are Very Up to Date) and Also Deleted Several Pages of Citations of Outdated Award-Winning OR Applications.

OTHER SPECIAL FEATURES OF THIS BOOK

- **An Emphasis on Real Applications.** The field of operations research is continuing to have a dramatic impact on the success of numerous companies and organizations around the world. Therefore, one of the goals of this book is to tell this story clearly and thereby excite students about the great relevance of the material they are studying. One way this goal is pursued is by including many realistic cases patterned after real applications at the end of chapters and on the book's website. Another way is the inclusion of many application vignettes scattered throughout the book that describe in a few paragraphs how an actual award-winning application of operations research had a powerful impact on a company or organization by using techniques like those studied in that portion of the book. For each application vignette, a problem also is included in the problems section of that chapter that requires the student to read the full article describing the application and then answer some questions. (The only application vignette that lacks this full article is the one in Chapter 1.) The next bullet point describes how students have immediate access to these articles.
- **Links to Many Articles Describing Dramatic OR Applications.** We are excited about a partnership with The Institute for Operations Research and the Management Sciences (INFORMS), our field's preeminent professional society, to provide a link on this book's website to each of the articles that fully describes the application that is summarized in one of the application vignettes. All of these articles appeared in an INFORMS journal called *Interfaces* (now retitled *INFORMS Journal on Applied Analytics* starting in 2019). (Information about INFORMS journals, meetings, job bank, scholarships, awards, and teaching materials is at www.informs.org.) These articles and the corresponding end-of-chapter problems provide instructors with the option of having their students delve into real applications that dramatically demonstrate the relevance of the material being covered in the lectures. It would even be possible to devote significant course time to discussing real applications.
- **A Wealth of Supplementary Chapters and Sections on the Website.** In addition to the nearly 1,000 pages in this book, another several hundred pages of supplementary material also are provided on this book's website (as outlined in the table of contents). This includes eight complete chapters, 12 supplements to chapters in the book, and dozens of additional cases. Most of the supplementary chapters include problems and selected references. Most of the supplements to chapters also have

problems. Today, when students think nothing of accessing material electronically, instructors should feel free to include some of this supplementary material in their courses.

- **Many Additional Examples Are Available.** An especially important learning aid on the book's website is a set of Solved Examples for almost every chapter in the book. We believe that most students will find the examples in the book fully adequate but that others will feel the need to go through additional examples. These solved examples on the website will provide the latter category of students the needed help, but without interrupting the flow of the material in the book on those many occasions when most students don't need to see an additional example. Many students also might find these additional examples helpful when preparing for an examination. We recommend to instructors that they point out this important learning aid to their students.
- **Great Flexibility for What to Emphasize.** We have found that there is great variability in what instructors want to emphasize in an introductory OR survey course. They might want to emphasize the mathematics and algorithms of operations research. Others will emphasize model formulation with little concern for the details of the algorithms needed to solve these models. Others want an even more applied course, with emphasis on applications and the role of OR in managerial decision making. Some instructors will focus on the deterministic models of OR, while others will emphasize stochastic models. There also are great differences in the kind of software (if any) that instructors want their students to use. All of this helps to explain why the book is a relatively large one. We believe that we have provided enough material to meet the needs of all of these kinds of instructors. Furthermore, the book is organized in such a way that it is relatively easy to pick and choose the desired material without loss of continuity. It even is possible to provide great flexibility on the kind of software (if any) that instructors want their students to use, as described below in the section on software options.
- **A Customizable Version of the Text Also is Available.** Because the text provides great flexibility for what to emphasize, an instructor can easily pick and choose just certain portions of the book to cover. Rather than covering most of the pages in the book, perhaps you wish to use only a much smaller portion of the text. Fortunately, McGraw-Hill provides an option for using a considerably smaller and less expensive version of the book that is customized to meet your needs. With McGraw-Hill Create™, you can include only the chapters you want to cover. You also can easily rearrange chapters, combine material from other content sources, and quickly upload content you have written, like your course syllabus or teaching notes. If desired, you can use Create to search for useful supplementary material in various other leading McGraw-Hill textbooks. For example, if you wish to emphasize spreadsheet modeling and applications, we would recommend including some chapters from the Hillier-Hillier textbook, *Introduction to Management Science: A Modeling and Case Studies Approach with Spreadsheets*. (That textbook includes a complete coverage of the Analytic Solver Platform for Education software package that has been dropped in this edition.) Arrange your book to fit your teaching style. Create even allows you to personalize your book's appearance by selecting the cover and adding your name, school, and course information. Order a Create book and you'll receive a complimentary print review copy in 3–5 business days or a complimentary electronic review copy (eComp) via e-mail in minutes. You can go to www.mcgrawhillcreate.com and register to experience how McGraw-Hill Create empowers you to teach your students your way.

A WEALTH OF SOFTWARE OPTIONS

A wealth of software options is provided on the book's website www.mhhe.com/hillier11e as outlined below:

- Excel spreadsheets: state-of-the-art spreadsheet formulations in Excel files for all relevant examples throughout the book. The standard Excel Solver can solve all of these examples.
- A number of Excel templates for solving basic models.
- Student versions of LINDO (a traditional optimizer) and LINGO (a popular algebraic modeling language), along with formulations and solutions for all relevant examples throughout the book.
- Student versions of MPL (a leading algebraic modeling language) along with an MPL Tutorial and MPL formulations and solutions for all relevant examples throughout the book.
- Student versions of several elite MPL solvers for linear programming, integer programming, convex programming, global optimization, etc.
- Queueing Simulator (for the simulation of queueing systems).
- OR Tutor for illustrating various algorithms in action.
- Interactive Operations Research (IOR) Tutorial for efficiently learning and executing algorithms interactively, implemented in Java 2 in order to be platform independent.

Numerous students have found OR Tutor and IOR Tutorial very helpful for learning various OR algorithms. When moving to the next stage of solving OR models automatically, surveys have found instructors almost equally split in preferring one of the following options for their students' use: (1) Excel spreadsheets, including Excel's Solver, (2) convenient traditional software (LINDO and LINGO), and (3) other state-of-the-art OR software (MPL and its elite solvers). For this edition, therefore, I have retained the philosophy of the last few editions of providing enough introduction in the book to enable the basic use of any of the three options without distracting those using another, while also providing ample supporting material for each option on the book's website.

There are only two software packages that accompanied the 10th edition that are not continued with this new edition. One is the Analytic Solver Platform for Education (ASPE) previously discussed in Sec. 3.5 and several subsequent places. The other is the TreePlan software for decision trees that was described in a supplement to Chapter 16. Our policy is that students must be able to use all the software provided with the book for their course work without any additional charge, but the owners of these two packages now are charging students for their use.

Additional Online Resources

- A *glossary* for every book chapter.
- *Data files* for various cases to enable students to focus on analysis rather than inputting large data sets.
- A *test bank* featuring moderately difficult questions that require students to show their work is being provided to instructors. Many of the questions in this test bank have previously been used successfully as test questions by the authors.
- A *solutions manual* and *image files* for instructors.

THE USE OF THE BOOK

The overall thrust of all the revision efforts has been to build upon the strengths of previous editions to more fully meet the needs of today's students. These revisions make the book even more suitable for use in a modern course that reflects contemporary practice in the field. The use of software is integral to the practice of operations research, so the wealth of software options accompanying the book provides great flexibility to the instructor in choosing the preferred types of software for student use. All the educational resources accompanying the book further enhance the learning experience. Therefore, the book and its website should fit a course where the instructor wants the students to have a single self-contained textbook that complements and supports what happens in the classroom.

The McGraw-Hill editorial team and I think that the net effect of the revision has been to make this edition even more of a "student's book"—clear, interesting, and well-organized with lots of helpful examples and illustrations, good motivation and perspective, easy-to-find important material, and enjoyable homework, without too much notation, terminology, and dense mathematics. We believe and trust that the numerous instructors who have used previous editions will agree that this is the best edition yet.

The prerequisites for a course using this book can be relatively modest. As with previous editions, the mathematics has been kept at a relatively elementary level. Most of Chaps. 1 to 15 (introduction, linear programming, and mathematical programming) require no mathematics beyond high school algebra. Calculus is used only in Chap. 13 (Nonlinear Programming) and in one example in Chap. 11 (Dynamic Programming). Matrix notation is used in Chap. 5 (The Theory of the Simplex Method), Chap. 6 (Duality Theory), Chap. 7 (Linear Programming under Uncertainty), Sec. 8.4 (An Interior-Point Algorithm), and Chap. 13, but the only background needed for this is presented in Appendix 4. For Chaps. 16 to 20 (probabilistic models), a previous introduction to probability theory is assumed, and calculus is used in a few places. In general terms, the mathematical maturity that a student achieves through taking an elementary calculus course is useful throughout Chaps. 16 to 20 and for the more advanced material in the preceding chapters.

The content of the book is aimed largely at the upper-division undergraduate level (including well-prepared sophomores) and at first-year (master's level) graduate students. Because of the book's great flexibility, there are many ways to package the material into a course. Chapters 1 and 2 give an introduction to the subject of operations research. Chapters 3 to 15 (on linear programming and mathematical programming) may essentially be covered independently of Chaps. 16 to 20 (on probabilistic models), and vice-versa. Furthermore, the individual chapters among Chaps. 3 to 15 are almost independent, except that they all use basic material presented in Chap. 3 and perhaps in Chap. 4. Parts of Chapters 5-8 are a little more challenging mathematically than the prior chapters. Chapters 6 and 7 and Sec. 8.2 draw upon Chap. 5. Sections 8.1 and 8.2 use parts of Chaps. 6 and 7. Section 10.6 assumes an acquaintance with the problem formulations in Secs. 9.1 and 9.3, while prior exposure to Secs. 8.3 and 9.2 is helpful (but not essential) in Sec. 10.7. Within Chaps. 16 to 20, there is considerable flexibility of coverage, although some integration of the material is available.

An elementary survey course covering linear programming, mathematical programming, and some probabilistic models can be presented in a quarter (40 hours) or semester by selectively drawing from material throughout the book. For example, a good survey of the field can be obtained from Chaps. 1, 2, 3, 4, 16, 17, 18, and 20, along

with parts of Chaps. 10 to 14. A more extensive elementary survey course can be completed in two quarters (60 to 80 hours) by excluding just a few chapters, for example, Chaps. 8, 15, and 19. Chapters 1 to 9 (and perhaps part of Chap. 10) form an excellent basis for a (one-quarter) course in linear programming. The material in Chaps. 10 to 15 covers topics for another (one-quarter) course in other deterministic models. Finally, the material in Chaps. 16 to 20 covers the probabilistic (stochastic) models of operations research suitable for presentation in a (one-quarter) course. In fact, these latter three courses (the material in the entire text) can be viewed as a basic one-year sequence in the techniques of operations research, forming the core of a master's degree program.

The book's website will provide any updates about the book, including an errata. To access this site, visit www.mhhe.com/hillier11e.

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I am indebted to an excellent group of reviewers who provided sage advice for the revision process. This group included

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In addition, thanks go to those instructors and students who sent email messages to provide their feedback on the 10th edition. Special thanks go to Andrew Denard, a student who found a considerable number of typos for me.

I am particularly grateful to three friends who provided expert advice on specific topics for this edition. I have known all of them well since they were students (and eventually PhD graduates) at Stanford a few decades ago, and all three have gone on to illustrious careers in the field. One is Irv Lustig, who currently is an Optimization principal with Princeton Consultants. Irv is well known as being on the leading edge of current developments at the interface between theory and practice, including in the area of analytics. A second is Vijay Mehrotra, a faculty member at the University of San Francisco who is a regular columnist for the *Analytics* magazine. The third is Edward Rothberg, who is the CEO and a leading computational scientist for GUROBI, a particularly prominent OR software company. Irv and Vijay guided me through the process of developing the four new up-to-date sections on analytics in the first two chapters. Ed identified the current state of the art for me regarding the factors affecting the speed of the simplex method (and its variants), as well as the factors affecting the relative performance of the simplex method and interior-point algorithms. This provided authoritative updates for Sections 4.10 and 4.11.

I also am very fortunate to have a strong team who contributed to recent editions in ways that supported the current edition as well. Our case writers, Karl Schmedders and Molly Stephens (both graduates of our department), wrote 24 elaborate cases for the 7th edition, and all of these cases continue to accompany this new edition. One of our department's former PhD students, Michael O'Sullivan, developed OR Tutor for the 7th edition (and continued here), based on part of the software that my son Mark Hillier had developed for the 5th and 6th editions. Mark (who was born the same year as the first edition, earned his PhD at Stanford, and now is a tenured Associate Professor of

Quantitative Methods at the University of Washington) provided both the spreadsheets and the Excel files (including many Excel templates) once again for this edition, as well as the Queueing Simulator. He also contributed greatly to Chap. 21 on the book's website. In addition, he updated both the 10th edition and the current 11th edition versions of the solutions manual. Earlier editions of this solutions manual were prepared in an exemplary manner by a long sequence of PhD students from our department, including Che-Lin Su for the 8th edition and Pelin Canbolat for the 9th edition. Che-Lin and Pelin did outstanding work that nicely paved the way for Mark's work on the solutions manual. Last, but definitely not least, my dear wife, Ann Hillier (another Stanford graduate with a minor in operations research), provided me with important help on a regular basis. All the individuals named above were vital members of the team.

I also owe a great debt of gratitude to three individuals and their companies for providing the special software and related information for the book. Another Stanford PhD graduate, William Sun (CEO of the software company Accelet Corporation), and his team did a brilliant job of starting with much of Mark Hillier's earlier software and implementing it anew in Java 2 as IOR Tutorial for the 7th edition, as well as further enhancing IOR Tutorial for the subsequent editions. Linus Schrage of the University of Chicago and the head of LINDO Systems (and my former faculty colleague at Stanford) has again provided LINGO and LINDO for the book's website. He also supervised the further development of LINGO/LINDO files for the various chapters as well as providing tutorial material for the book's website. Another long-time friend, Bjarni Kristjansson (who heads Maximal Software), did the same thing for the MPL/Solvers files and MPL tutorial material, as well as arranging to provide a student version of MPL and various elite solvers for the book's website. These three individuals and their companies—Accelet Corporation, LINDO Systems, and Maximal Software—have made an invaluable contribution to this book.

I also am excited about the partnership with INFORMS that began with the 9th edition. Students can benefit greatly by reading about top-quality applications of operations research. This preeminent professional OR society is enabling this by providing a link to the articles in *Interfaces* (now called *INFORMS Journal on Applied Analytics*) that describe the applications of OR that are summarized in the application vignettes provided in the book.

It was a real pleasure working with McGraw-Hill's thoroughly professional editorial and production staff, including Theresa Collins (the Product Developer during most of the development of this edition), and Jason Stauter (Content Project Manager).

Just as so many individuals made important contributions to this edition, I would like to invite each of you to start contributing to the next edition by using my email address below to send me your comments, suggestions, and errata to help me improve the book in the future. In giving my email address, let me also assure instructors that I will continue to follow the policy of not providing solutions to problems and cases in the book to anybody (including your students) who contacts me.

Enjoy the book.

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CHAPTER 1

Introduction

1.1 THE ORIGINS OF OPERATIONS RESEARCH

Since the advent of the industrial revolution, the world has seen a remarkable growth in the size and complexity of organizations. The artisans' small shops of an earlier era have evolved into the billion-dollar corporations of today. An integral part of this revolutionary change has been a tremendous increase in the division of labor and segmentation of management responsibilities in these organizations. The results have been spectacular. However, along with its blessings, this increasing specialization has created new problems, problems that are still occurring in many organizations. One problem is a tendency for the many components of an organization to grow into relatively autonomous empires with their own goals and value systems, thereby losing sight of how their activities and objectives mesh with those of the overall organization. What is best for one component frequently is detrimental to another, so the components may end up working at cross purposes. A related problem is that as the complexity and specialization in an organization increase, it becomes more and more difficult to allocate the available resources to the various activities in a way that is most effective for the organization as a whole. These kinds of problems and the need to find a better way to solve them provided the environment for the emergence of **operations research** (commonly referred to as **OR**).

The roots of OR can be traced back many decades,[1] when early attempts were made to use a scientific approach in the management of organizations. However, the beginning of the activity called *operations research* has generally been attributed to the military services early in World War II. Because of the war effort, there was an urgent need to allocate scarce resources to the various military operations and to the activities within each operation in an effective manner. Therefore, the British and then the U.S. military

[1]Selected Reference 7 (cited at the end of the chapter) provides an entertaining history of operations research that traces its roots as far back as 1564 by describing a considerable number of scientific contributions from 1564 to 2004 that influenced the subsequent development of OR. Also see Selected References 1 and 6 for further details about this history. For example, Chapter 10 in Selected Reference 1 tells the interesting story about how in 1939 a Russian mathematician and economist, Leonid Kantoravich, published a very important OR paper, "Mathematical Methods of Organizing and Planning Production," in Russian (later translated into English in *Management Science,* **6**(4): 366–422, July 1960). Unknown in the West until considerably later, this paper (and a couple of his related papers) developed fundamental results in a key OR area (linear programming). Kantorovich was awarded the Nobel Prize in Economics in 1975 mainly for this work.

management called upon a large number of scientists to apply a scientific approach to dealing with this and other strategic and tactical problems. In effect, they were asked to do *research on* (military) *operations.* These teams of scientists sometimes were called *operations research teams* (or *OR teams* for short). By developing effective methods of using the new tool of radar, these teams were instrumental in winning the Air Battle of Britain. Through their research on how to better manage convoy and antisubmarine operations, they also played a major role in winning the Battle of the North Atlantic. Similar efforts assisted the Island Campaign in the Pacific.

When the war ended, the success of OR in the war effort spurred interest in applying OR outside the military as well. As the industrial boom following the war was running its course, the problems caused by the increasing complexity and specialization in organizations were again coming to the forefront. It was becoming apparent to a growing number of people, including business consultants who had served on or with the OR teams during the war, that these were basically the same problems that had been faced by the military but in a different context. By the early 1950s, these individuals had introduced the use of OR to a variety of organizations in business, industry, and government. The rapid spread of OR soon followed. (Selected Reference 1 at the end of the chapter recounts the development of the field of operations research by describing the lives and contributions of 43 OR pioneers.)

At least two other factors that played a key role in the rapid growth of OR during this period can be identified. One was the substantial progress that was made early in improving the techniques of OR. After the war, many of the scientists who had participated on OR teams or who had heard about this work were motivated to pursue research relevant to the field; important advancements in the state of the art resulted. A prime example is the *simplex method* for solving linear programming problems, developed by George Dantzig in 1947. Some of the standard tools of OR, such as linear programming, dynamic programming, queueing theory, and inventory theory, were relatively well developed before the end of the 1950s (although extensive research into both these tools and many new OR techniques have continued to the present day).

A second factor that gave great impetus to the growth of the field was the onslaught of the *computer revolution.* A large amount of computation is usually required to deal most effectively with the complex problems typically considered by OR. Doing this by hand would often be out of the question. Therefore, the development of electronic digital computers, with their ability to perform arithmetic calculations millions of times faster than a human being, was a tremendous boon to OR. A further boost came in the 1980s with the development of increasingly powerful personal computers accompanied by good software packages for doing OR. This brought the use of OR within the easy reach of much larger numbers of people, and this progress further accelerated in the 1990s and into the 21st century. For example, the widely used spreadsheet package, Microsoft Excel, provides a Solver that will solve a variety of OR problems. Today, literally millions of individuals have ready access to OR software. Consequently, a whole range of computers now are routinely being used to solve OR problems, including some of enormous size.

This ongoing acceleration of computer power has continued to contribute to the growth of OR even up to the present time. The field continues to make substantial progress in further developing the power of its methodology, which has led to undertaking exceptionally ambitious applications. OR today is far ahead of where it was even one or two decades ago. For example, Sec. 1.3 describes the exciting story of how the OR discipline has embraced the powerful new discipline of *analytics* (sometimes called *data science*) as an approach to decision making that largely overlaps and further enriches the OR approach. Analytics was still in its infancy as recently as 2006. Section 1.4 introduces

some of the dramatic applications of OR that have been taking place in recent years, including such nontraditional applications as the eradication of polio, increasing the world's food production, and combating cancer. Many of these applications will be further described elsewhere in the book and links often will be provided to articles presenting further details. Section 1.5 describes some trends that should further increase the future impact of operations research.

1.2 THE NATURE OF OPERATIONS RESEARCH

As its name implies, operations research involves "research on operations." Thus, operations research is applied to problems that concern how to conduct and coordinate the *operations* (i.e., the *activities*) within an organization. The nature of the organization is essentially immaterial, and in fact, OR has been applied extensively in such diverse areas as manufacturing, transportation, construction, telecommunications, financial planning, health care, the military, and public services, to name just a few. Therefore, the breadth of application is unusually wide.

The *research* part of the name means that operations research uses an approach that resembles the way research is conducted in established scientific fields. To a considerable extent, the *scientific method* is used to investigate the problem of concern. (In fact, the term *management science* sometimes is used as a synonym for operations research.) In particular, the process begins by carefully observing and formulating the problem, including gathering all relevant data and using it to better understand the problem and what lies ahead. The next step in the scientific method is to construct a scientific (typically mathematical) model that attempts to abstract the essence of the real problem. It is then hypothesized that this model is a sufficiently precise representation of the essential features of the situation that the conclusions (solutions) obtained from the model are also valid for the real problem. Next, suitable experiments are conducted to test this hypothesis, modify it as needed, and eventually verify some form of the hypothesis. (This step is frequently referred to as *model validation.*) Thus, in a certain sense, operations research involves creative scientific research into the fundamental properties of operations. However, there is more to it than this. Specifically, OR is also concerned with the practical management of the organization. Therefore, to be successful, OR must also provide positive, understandable conclusions to the decision maker(s) when they are needed.

Still another characteristic of OR is its broad viewpoint. As implied in the preceding section, OR adopts an organizational point of view. Thus, it attempts to resolve the conflicts of interest among the components of the organization in a way that is best for the organization as a whole. This does not imply that the study of each problem must give explicit consideration to all aspects of the organization; rather, the objectives being sought must be consistent with those of the overall organization.

An additional characteristic is that OR frequently attempts to search for a *best* solution (referred to as an *optimal* solution) for the model that represents the problem under consideration. (We say *a* best instead of *the* best solution because multiple solutions may be tied as best.) Rather than simply improving the status quo, the goal is to identify a best possible course of action. Although it must be interpreted carefully in terms of the practical needs of management, this "search for optimality" is an important theme in OR.

All these characteristics lead quite naturally to still another one. It is evident that no single individual should be expected to be an expert on all the many aspects of OR work or the problems typically considered; this would require a group of individuals having diverse backgrounds and skills. Therefore, when a full-fledged OR study of a new problem is undertaken, it is usually necessary to use a *team approach.* Such an OR team typically

needs to include individuals who collectively are highly trained in mathematics, statistics and probability theory, data science, economics, business administration, computer science, engineering, the physical sciences, and the behavioral sciences, as well as the special techniques of OR. The team also needs to have the necessary experience and variety of skills to give appropriate consideration to the many ramifications of the problem throughout the organization.

1.3 THE RELATIONSHIP BETWEEN ANALYTICS AND OPERATIONS RESEARCH

There has been great buzz throughout the business world in recent years about something called **analytics** (or *business analytics*) and the importance of incorporating analytics into managerial decision making. The primary impetus for this buzz was a series of articles and books by Thomas H. Davenport, a renowned thought-leader who has helped hundreds of companies worldwide to revitalize their business practices. He initially introduced the concept of analytics in the January 2006 issue of the *Harvard Business Review* with an article, "Competing on Analytics," that now has been named as one of the 10 must-read articles in the magazine's 90-year history. This article was soon followed by two best-selling books entitled *Competing on Analytics: The New Science of Winning* and *Analytics at Work: Smarter Decisions, Better Results.* (See Selected References 2 and 3 at the end of the chapter for the citations, where the former reference is a new edition of the 2007 landmark book that first introduced business leaders to analytics.)

So what is analytics? In contrast to operations research, analytics is not a single discipline with its own well-defined body of techniques. Analytics instead includes all the *quantitative decision sciences*. Traditional types of quantitative decision sciences include mathematics, statistics, computer science, and operations research, but other types of quantitative decision sciences also arise in such areas as information technology, business analysis, industrial engineering, management science, etc. Another major component of analytics is what is now called **data science**, which itself draws heavily on statistics and computer science to make sense of what may be vast amounts of data while also exploiting an explosion in computational capability.

Thus, any application of analytics draws on any of the quantitative decision sciences that can be helpful in analyzing a given problem. Therefore, a company's analytics group might include members with titles such as mathematician, statistician, computer scientist, data scientist, information technologist, business analyst, industrial engineer, management scientist, and operations research analyst.

At this point, the members of an operations research group might object, saying that their operations research studies often draw upon these other quantitative decision sciences as well. This frequently is true, but it also is true that many applications of analytics draw *mainly* from certain other quantitative decision sciences *instead* of operations research. This often occurs when the issue being addressed is to try to gain insights from all the available data, so that *data science* and *statistics* become the key quantitative decision sciences.

Analytics has grown in prominence over the past decade largely because we have entered into the era of *big data* where massive amounts of data (accompanied by massive amounts of computational power) are now commonly available to many businesses and organizations to help guide managerial decision making. The current data surge is coming from sophisticated computer tracking of shipments, sales, suppliers, and customers, as well as email, web traffic, social networks, images, and video. A primary focus of analytics is on how to make the most effective use of all these data.

The application of analytics can be divided into three overlapping categories. The traditional names and brief descriptions of these categories follow:

Category 1: **Descriptive analytics** (analyzing data to create informative descriptions of what has happened in the past or is happening in the present).
Category 2: **Predictive analytics** (using models to create predictions of what is likely to happen in the future).
Category 3: **Prescriptive analytics** (using decision models, including optimization models, to create and/or advise managerial decision making).

The first of these categories, *descriptive analytics*, requires dealing with perhaps massive amounts of data. Information technology is used to store and access the data on what has happened in the past, as well as to record what is happening now. Descriptive analytics then uses innovative techniques to locate the relevant data and identify the interesting patterns and summary data in order to better describe and understand what has been happening in the past or what is happening now. *Data mining* is one important technique that is available for doing this (and is used even more extensively for performing predictive analytics). *Business intelligence* is another name that is sometimes used for this category (and category 2 as well). (We will further describe how descriptive analytics deal with big data in Sec. 2.3.)

Predictive analytics involves applying statistical models to predict future events or trends. The models underlying forecasting methods, such as those described in Chapter 27 (one of the supplementary chapters on this book's website), sometimes are used here. Simulation (Chapter 20) also can be useful for demonstrating future events that can occur. However, in addition to data mining, a variety of other important data-based techniques also are used to predict future events or trends. Some highly trained analytics professionals who specialize largely in using these techniques are called **data scientists**. (We will describe these techniques further in Sec. 2.4.) Because some of the methods of predictive analytics are quite sophisticated, this category tends to be more advanced than the first one.

Prescriptive analytics is the final (and most advanced) category. It involves applying decision models to the data to prescribe what should be done in the future. The powerful techniques of operations research described in many of the chapters of this book (including a wide variety of decision models and algorithms for finding optimal solutions) generally are used here. The purpose is to guide managerial decision making.

Having introduced some of the basic traditional terminology of analytics (descriptive, predictive, and prescriptive analytics, data science, data scientist, etc.), we should point out that the terminology in this young area continues to change at a rapid pace due to innovation and lots of market activity. For example, analytics now is sometimes referred to as *data science*. Additional changes in terminology probably lie ahead.

Operations research analysts often deal with all three of the categories of analytics described above. OR analysts need to perform some descriptive analytics to gain some understanding of the data. They also frequently need to perform some predictive analytics, perhaps by using standard statistics techniques such as forecasting methods or standard OR techniques such as simulation, to gain some understanding of what is likely to happen in the future. OR analysts have special expertise for performing prescriptive analytics by applying powerful OR techniques.

When comparing OR analysts and analytics professionals, it is the analytics professionals who typically have extra expertise in the descriptive analytics area. They also have a larger toolkit when dealing with data preparation and predictive analytics, although OR analysts typically also have some expertise in these areas. OR analysts normally take the lead when performing prescriptive analytics. Therefore, when conducting a full-fledged study that requires performing all three types of analytics, an ideal team would

include analytics professionals, data scientists, and OR analysts. Looking to the future, the two approaches should tend to merge somewhat over time as these types of professionals learn from each other.

Although analytics was initially introduced as a key tool for mainly business organizations, it also can be a powerful tool in other contexts. As one example, analytics (including operations research) played a key role in the 2012 presidential campaign in the United States. The Obama campaign management hired a multidisciplinary team of statisticians, predictive modelers, data-mining experts, mathematicians, software programmers, and operations research analysts. It eventually built an entire analytics department five times as large as that of its 2008 campaign. With all this analytics input, the Obama team launched a full-scale and all-front campaign, leveraging massive amounts of data from various sources to directly micro-target potential voters and donors with tailored messages. The election had been expected to be a very close one, but the Obama "ground game" that had been propelled by descriptive and predictive analytics was given much of the credit for the clear-cut Obama win.

Because of the key contribution of analytics to the Democratic presidential campaign in 2012, major political campaigns in later years have continued to make heavy use of analytics. This certainly was true for the Democratic presidential campaign in 2016. However, this use of analytics was met this time by a controversial use of analytics on the Republican side that was perhaps even more effective. In particular, a foreign organization called Cambridge Analytica misled and exploited Facebook to gather politically useful data from over 70 million potential American voters. This information was then used to send tailored messages to these potential voters to encourage them to vote for the Republican ticket. (The exposure many months later of what they had done led to Cambridge Analytica shutting down soon thereafter.)

Another famous application of analytics is described in the book *Moneyball* (see Selected Reference 10) and a subsequent 2011 movie with the same name that is based on this book. They tell the true story of how the Oakland Athletics baseball team achieved great success, despite having one of the smallest budgets in the major leagues, by using various kinds of nontraditional data (referred to as *sabermetrics*) to better evaluate the potential of players available through a trade or the draft. Although these evaluations often flew in the face of conventional baseball wisdom, both descriptive analytics and predictive analytics were being used to identify overlooked players who could greatly help the team. After witnessing the impact of analytics, all major league baseball teams now have hired analytics professionals, and analytics also is spreading down into the minor leagues.

In fact, substantial use of **sports analytics** also has been spreading to various other kinds of sports teams as well. As far back as 2012, Selected Reference 4 devoted an entire special issue of *Interfaces* to the application of sports analytics to such sports as hockey, golf, and motorcycle racing, as well as baseball. Selected Reference 5 is another special issue published in 2012 that is devoted to the application of analytics to sports scheduling.

One special success story in the area of sports analytics involves the Golden State Warriors, the most successful professional basketball team in the NBA (National Basketball Association) over a span of a few years beginning with the 2014–2015 season. Leading up to this success, a young Stanford graduate by the name of Kirk Lacob (the son of the team's general manager) oversaw a pioneering program to bring analytics (including machine learning and data science) to the basketball court in a major way. Analytics was used to guide both personnel decisions and the selection of strategies on the court. Some months after they won the 2015 NBA championship, the Golden State Warriors won the Best Analytics Organization award at the MIT Sloan Sports Analytics

Conference in March 2016. A later book entitled *Betaball: How Silicon Valley and Science Built One of the Greatest Basketball Teams in History* (see Selected Reference 12) further expands on this special success story from a broader viewpoint.

To avoid being left behind, professional football teams in the NFL (National Football League) also are adopting analytics. For example, the Philadelphia Eagles were heavy users of analytics in going all the way to winning the 2018 Super Bowl.

In addition to the usage of analytics in the political and sports worlds, there are many other areas where analytics is having a real impact. Examples of such areas include healthcare, combating crime, and financial analysis, among others. However, the greatest usage by far now is occurring in the business world. Indeed, analytics sometimes is called **business analytics** because business applications are so prevalent.

After a slow start, the top management of numerous business organizations now understands the impact that analytics can have on the bottom line and they are very interested in increasing the role of the analytics group in their organization. This will require many more people trained in analytics and operations research. As far back as 2011, a report (Selected Reference 13) from the McKinsey Global Institute (the research arm of the prestigious management consulting firm McKinsey & Company) predicted that by 2018 the United States could have a shortage of 140,000 to 190,000 people with deep analytical skills. A second prediction was for a shortage of 1.5 million managers and analysts with the experience and expertise to use the analysis of big data to make effective and efficient decisions. A follow-up report from the McKinsey Global Institute in December 2016 (Selected Reference 9) found that these kinds of shortages were indeed occurring. The report also highlighted a prediction that the rapidly increasing demand could lead to a shortage of about 250,000 data scientists.

Universities are now responding to this great need. There are now hundreds of schools in the United States or abroad that have, or have committed to launch, curriculum at the undergraduate and graduate levels with degrees or certificates in analytics. Courses that cover the material in this book would be a key component of these programs, along with courses that emphasize other areas of analytics (e.g., statistics and data mining).

This creates an outstanding opportunity for students with a STEM (Science, Technology, Engineering, and Mathematics) focus. In the words of the thought leader Thomas H. Davenport (who was introduced in the first paragraph of this section), the job of an *analytics professional* promises to be the "sexiest job in the 21st century." In 2016, 2017, and 2018, the job site *Glassdoor* also named *data scientist* as the best job in America. (The title of data scientist normally refers to an exceptionally talented and versatile professional who specializes in applying all aspects of data science, ranging from the cleaning and preparation of data to the writing of software for analyzing data and then to implementing various algorithms for performing analytics, including especially predictive analytics.)

Similar opportunities also are available for STEM students who become *operations research analysts. U.S. News and World Report* annually publishes a list of the best jobs in the United States, based on a variety of factors such as salary and job satisfaction. In recent years (2016–2018), their list of the top business jobs in the United States has consistently ranked *operations research analyst* (as well as *statistician* and *mathematician*) in the top 10. Indeed, *operations research analyst* was given the number 2 spot on the 2016 list. Furthermore, this profession ranks very high in terms of having a high percentage of women (similar to that for men) working in the profession. For example, *USA Today* on January 12, 2016, reported that 55.4 percent of all OR analysts employed in the United States at that time were women. Furthermore, the demand for both men and women in this field continues to grow rapidly. According to the U.S. Bureau of Labor Statistics in June 2018, the employment of OR analysts in the United States

continues to be projected to grow "much faster than the average for all occupations" over the next decade (2016–2026). The bureau indicates that the typical educational requirement for entry-level positions as an OR analyst is a bachelor's degree, but that some employers may prefer to hire applicants with a master's degree. OR analysts typically have a degree in business, operations research, management science, analytics, mathematics, engineering, computer science, or another technical or quantitative field. The median annual salary for OR analysts was $81,390 in May 2017.

When describing the relationship between analytics and operations research throughout this section, we have pointed out a number of differences in emphases. However, these distinctions should diminish over time. Analytics and operations research complement each other so well that each should gradually claim ownership of the special techniques of the other. This gradual merger should particularly benefit operations research as a field. The name *operations research* does a poor job of conveying what it is to individuals outside this field. By contrast, the name *analytics* is so much better recognized as having real value. Therefore, it seems likely that the name *analytics* gradually will be adopted to incorporate the traditional techniques of operations research as well. With the enthusiastic appreciation of what analytics has to offer, this also should increase the appreciation of these traditional techniques.

There is considerable evidence that the close partnership between analytics and operations research is continuing to deepen. For example, consider the initiatives started some years ago by **INFORMS** (the Institute for Operations Research and the Management Sciences), which is the largest professional society of OR academics, professionals, and students in the world. This organization holds a well-attended Business Analytics Conference annually in addition to the annual INFORMS Meeting that encompasses both OR and analytics. INFORMS publishes 16 prestigious journals in various areas of operations research and analytics. One of the most popular is the *INFORMS Journal on Applied Analytics* (previously entitled *Interfaces* before 2019*)*. This journal features articles describing dramatic applications that exploit the close relationship between analytics and operations research. Another INFORMS publication is the *Analytics Magazine,* which is published six times per year to focus on important developments in the analytics world. INFORMS includes some special-interest societies within it and a particularly large one is its Analytics Society. In addition, INFORMS manages the Certified Analytics Professional (CAP) program, which certifies analytics professionals only after they meet certain experience and education requirements and then pass a rigorous test. (An Associate Certified Analytics Professional designation also is available for qualified entry-level analytics professionals who pass a test.) In all these ways, this prestigious OR society has embraced analytics as a vital complement to the traditional tools of operations research.

The momentum of the analytics movement is indeed continuing to rapidly grow. Because operations research is at the core of advanced analytics, the usage of the powerful techniques of operations research introduced in this book also should continue to grow rapidly. However, without even looking to the future, the impact of operations research over past years has also been impressive, as described in the next section.

1.4 THE IMPACT OF OPERATIONS RESEARCH

Operations research has had an impressive impact on improving the efficiency of numerous organizations around the world. In the process, OR has made a significant contribution to increasing the productivity of the economies of various countries. There now are a few dozen member countries in the International Federation of Operational Research Societies

An Application Vignette

General Motors (GM) is one of the largest and most successful companies in the world. One major reason for this great success is that GM also is one of the world's leading users of advanced analytics and operations research. In recognition of the great impact that the application of these techniques has had on the success of the company, GM was awarded the **2016 INFORMS Prize**.

INFORMS (the Institute for Operations Research and the Management Sciences) awards the INFORMS Prize to just one organization each year for its particularly exceptional record in applying advanced analytics and operations research/management science (OR/MS) throughout the organization. The award winner must have repeatedly applied these techniques in pioneering, varied, novel, and lasting ways. Following is the citation that describes why GM won the prize for 2016.

Citation: The 2016 INFORMS Prize is awarded to General Motors for its sustained record of innovative and impactful applied operations research and advanced analytics.

General Motors has hundreds of OR/MS practitioners worldwide who play a vital role in driving data-driven decisions in everything from designing, building, selling, and servicing vehicles to purchasing, logistics, and quality. The team is constantly developing new business models and vetting emerging opportunities.

GM has developed new market research and analysis techniques to understand what products and features customers most want, to determine the ideal vehicles for their dealers to stock, and to identify the steps they can take to achieve GM's goal of creating customers for life.

GM is also leading the industry by using data science and advanced analytics to predict failure of automotive components and systems before customers are inconvenienced. GM's industry-first Proactive Alert messages notify customers through their OnStar system of a possible malfunction, transforming a potential emergency repair into routine planned maintenance.

"Over the last seven decades, OR/MS techniques have been used to improve our understanding of everything from traffic science and supply chain logistics to manufacturing productivity, product development, vehicles telematics and prognostics," said Gary Smyth, executive director of GM Global R&D Laboratories. "These approaches to problem solving permeate almost everything we do."

The impact OR/MS is now having on its business accelerated in 2007, when GM created a center of expertise for Operations Research to promote best practices and transfer new technologies. It since has expanded to include partner teams in product development, supply chain, finance, information technology, and other functions.

"General Motors: Past Awards 2016 INFORMS Prize: Winner(s)," *Informs*. Accessed March 25, 2019, https://www.informs.org/

(IFORS), with each country having a national OR society. Both Europe and Asia have federations of OR societies to coordinate holding international conferences and publishing international journals in those continents. In addition, the Institute for Operations Research and the Management Sciences (**INFORMS**) is an international OR society that is headquartered in the United States. As mentioned in the preceding section, INFORMS publishes many of the leading journals in the field, including one called *INFORMS Journal on Applied Analytics* that regularly publishes articles describing major OR studies and the impact they had on their organizations. (Prior to 2019, this journal was called *Interfaces*.)

To give you a better notion of the wide applicability of OR, we list some actual applications in Table 1.1 that have been described in this INFORMS journal. Many of these applications were winners or finalists in the prestigious international competition sponsored by INFORMS to identify the most significant OR application of the year. (The name of the prize now is the *Franz Edelman Award for Achievement in Advanced Analytics, Operations Research and the Management Sciences,* where *Advanced Analytics* was added to the name in 2019.) Note the diversity of organizations and applications in the first two columns of Table 1.1. The third column identifies the section where an "application vignette" devotes several paragraphs to describing the application and