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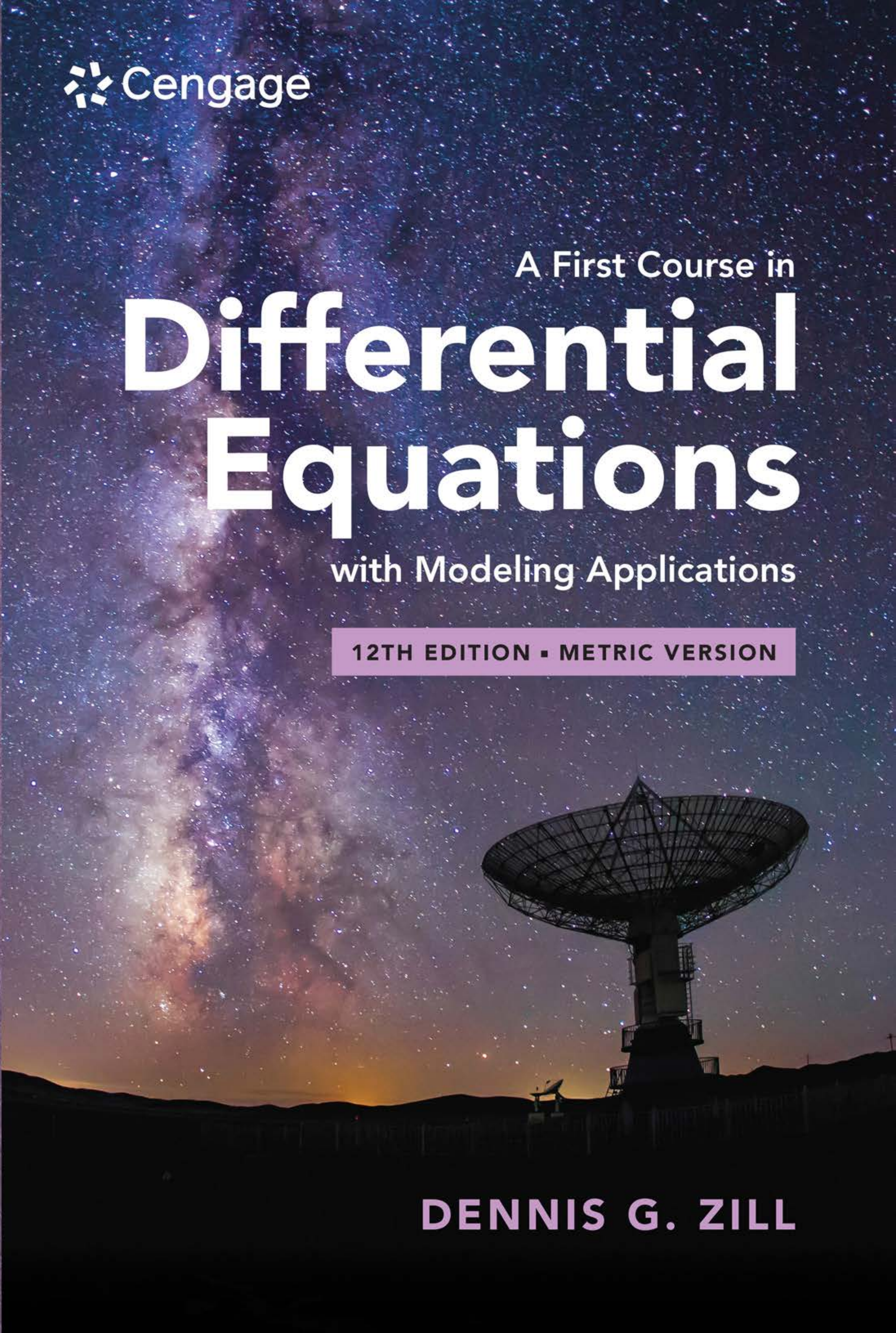
A First Course in Differential Equations with Modeling Applications

12TH EDITION ▪ METRIC VERSION

DENNIS G. ZILL



Review of Differentiation

Rules

1. **Constant:** $\dfrac{d}{dx}c = 0$

2. **Constant multiple:** $\dfrac{d}{dx}c\,f(x) = c\,f'(x)$

3. **Sum:** $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

4. **Product:** $\dfrac{d}{dx}f(x)g(x) = f(x)g'(x) + g(x)f'(x)$

5. **Quotient:** $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

6. **Chain:** $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

7. **Power:** $\dfrac{d}{dx}x^n = nx^{n-1}$

8. **Power:** $\dfrac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$

Functions

Trigonometric:

9. $\dfrac{d}{dx}\sin x = \cos x$

10. $\dfrac{d}{dx}\cos x = -\sin x$

11. $\dfrac{d}{dx}\tan x = \sec^2 x$

12. $\dfrac{d}{dx}\cot x = -\csc^2 x$

13. $\dfrac{d}{dx}\sec x = \sec x \tan x$

14. $\dfrac{d}{dx}\csc x = -\csc x \cot x$

Inverse trigonometric:

15. $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$

16. $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$

17. $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$

18. $\dfrac{d}{dx}\cot^{-1} x = -\dfrac{1}{1+x^2}$

19. $\dfrac{d}{dx}\sec^{-1} x = \dfrac{1}{|x|\sqrt{x^2-1}}$

20. $\dfrac{d}{dx}\csc^{-1} x = -\dfrac{1}{|x|\sqrt{x^2-1}}$

Hyperbolic:

21. $\dfrac{d}{dx}\sinh x = \cosh x$

22. $\dfrac{d}{dx}\cosh x = \sinh x$

23. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$

24. $\dfrac{d}{dx}\coth x = -\operatorname{csch}^2 x$

25. $\dfrac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$

26. $\dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$

Inverse hyperbolic:

27. $\dfrac{d}{dx}\sinh^{-1} x = \dfrac{1}{\sqrt{x^2+1}}$

28. $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$

29. $\dfrac{d}{dx}\tanh^{-1} x = \dfrac{1}{1-x^2}$

30. $\dfrac{d}{dx}\coth^{-1} x = \dfrac{1}{1-x^2}$

31. $\dfrac{d}{dx}\operatorname{sech}^{-1} x = -\dfrac{1}{x\sqrt{1-x^2}}$

32. $\dfrac{d}{dx}\operatorname{csch}^{-1} x = -\dfrac{1}{|x|\sqrt{x^2+1}}$

Exponential:

33. $\dfrac{d}{dx}e^x = e^x$

34. $\dfrac{d}{dx}a^x = a^x(\ln a)$

Logarithmic:

35. $\dfrac{d}{dx}\ln|x| = \dfrac{1}{x}$

36. $\dfrac{d}{dx}\log_a x = \dfrac{1}{x(\ln a)}$

Integral defined:

37. $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt = g(x)$

38. $\dfrac{d}{dx}\displaystyle\int_a^b g(x, t)\,dt = \int_a^b \frac{\partial}{\partial x}g(x, t)\,dt$

Brief Table of Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\int \frac{1}{u}\,du = \ln|u| + C$

3. $\int e^u\,du = e^u + C$

4. $\int a^u\,du = \frac{1}{\ln a}a^u + C$

5. $\int \sin u\,du = -\cos u + C$

6. $\int \cos u\,du = \sin u + C$

7. $\int \sec^2 u\,du = \tan u + C$

8. $\int \csc^2 u\,du = -\cot u + C$

9. $\int \sec u \tan u\,du = \sec u + C$

10. $\int \csc u \cot u\,du = -\csc u + C$

11. $\int \tan u\,du = -\ln|\cos u| + C$

12. $\int \cot u\,du = \ln|\sin u| + C$

13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

15. $\int u \sin u\,du = \sin u - u\cos u + C$

16. $\int u \cos u\,du = \cos u + u\sin u + C$

17. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

18. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

19. $\int \tan^2 u\,du = \tan u - u + C$

20. $\int \cot^2 u\,du = -\cot u - u + C$

21. $\int \sin^3 u\,du = -\frac{1}{3}\left(2 + \sin^2 u\right)\cos u + C$

22. $\int \cos^3 u\,du = \frac{1}{3}\left(2 + \cos^2 u\right)\sin u + C$

23. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

24. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

25. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

26. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

27. $\int \sin au \sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

28. $\int \cos au \cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

29. $\int e^{au} \sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

30. $\int e^{au} \cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

31. $\int \sinh u\,du = \cosh u + C$

32. $\int \cosh u\,du = \sinh u + C$

33. $\int \operatorname{sech}^2 u\,du = \tanh u + C$

34. $\int \operatorname{csch}^2 u\,du = -\coth u + C$

35. $\int \tanh u\,du = \ln(\cosh u) + C$

36. $\int \coth u\,du = \ln|\sinh u| + C$

37. $\int \ln u\,du = u\ln u - u + C$

38. $\int u \ln u\,du = \frac{1}{2}u^2 \ln u - \frac{1}{4}u^2 + C$

39. $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C$

40. $\int \frac{1}{\sqrt{a^2+u^2}}\,du = \ln\left|u + \sqrt{a^2+u^2}\right| + C$

41. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$

42. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u + \sqrt{a^2+u^2}\right| + C$

43. $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

44. $\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$

***Note*:** Some techniques of integration, such as integration by parts and partial fractions, are reviewed in the *Student Resource Manual* that accompanies this text.

Table of Laplace Transforms

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1. 1	$\dfrac{1}{s}$
2. t	$\dfrac{1}{s^2}$
3. t^n	$\dfrac{n!}{s^{n+1}}$, n a positive integer
4. $t^{-1/2}$	$\sqrt{\dfrac{\pi}{s}}$
5. $t^{1/2}$	$\dfrac{\sqrt{\pi}}{2s^{3/2}}$
6. t^α	$\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > -1$
7. $\sin kt$	$\dfrac{k}{s^2+k^2}$
8. $\cos kt$	$\dfrac{s}{s^2+k^2}$
9. $\sin^2 kt$	$\dfrac{2k^2}{s(s^2+4k^2)}$
10. $\cos^2 kt$	$\dfrac{s^2+2k^2}{s(s^2+4k^2)}$
11. e^{at}	$\dfrac{1}{s-a}$
12. $\sinh kt$	$\dfrac{k}{s^2-k^2}$
13. $\cosh kt$	$\dfrac{s}{s^2-k^2}$
14. $\sinh^2 kt$	$\dfrac{2k^2}{s(s^2-4k^2)}$
15. $\cosh^2 kt$	$\dfrac{s^2-2k^2}{s(s^2-4k^2)}$
16. te^{at}	$\dfrac{1}{(s-a)^2}$
17. $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$, n a positive integer
18. $e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
19. $e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
20. $e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
21. $e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$
22. $t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$
23. $t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$
24. $\sin kt + kt\cos kt$	$\dfrac{2ks^2}{(s^2+k^2)^2}$
25. $\sin kt - kt\cos kt$	$\dfrac{2k^3}{(s^2+k^2)^2}$
26. $t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$
27. $t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$
28. $\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$
29. $\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$
30. $1-\cos kt$	$\dfrac{k^2}{s(s^2+k^2)}$
31. $kt-\sin kt$	$\dfrac{k^3}{s^2(s^2+k^2)}$
32. $\dfrac{a\sin bt - b\sin at}{ab(a^2-b^2)}$	$\dfrac{1}{(s^2+a^2)(s^2+b^2)}$
33. $\dfrac{\cos bt - \cos at}{a^2-b^2}$	$\dfrac{s}{(s^2+a^2)(s^2+b^2)}$
34. $\sin kt\sinh kt$	$\dfrac{2k^2 s}{s^4+4k^4}$
35. $\sin kt\cosh kt$	$\dfrac{k(s^2+2k^2)}{s^4+4k^4}$
36. $\cos kt\sinh kt$	$\dfrac{k(s^2-2k^2)}{s^4+4k^4}$
37. $\cos kt\cosh kt$	$\dfrac{s^3}{s^4+4k^4}$
38. $\sin kt\cosh kt + \cos kt\sinh kt$	$\dfrac{2ks^2}{s^4+4k^4}$
39. $\sin kt\cosh kt - \cos kt\sinh kt$	$\dfrac{4k^3}{s^4+4k^4}$
40. $\sinh kt - \sin kt$	$\dfrac{2k^3}{s^4-k^4}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
41. $\cosh kt - \cos kt$	$\dfrac{2k^2 s}{s^4 - k^4}$
42. $J_0(kt)$	$\dfrac{1}{\sqrt{s^2 + k^2}}$
43. $\dfrac{e^{bt} - e^{at}}{t}$	$\ln \dfrac{s-a}{s-b}$
44. $\dfrac{2(1 - \cos kt)}{t}$	$\ln \dfrac{s^2 + k^2}{s^2}$
45. $\dfrac{2(1 - \cosh kt)}{t}$	$\ln \dfrac{s^2 - k^2}{s^2}$
46. $\dfrac{\sin at}{t}$	$\arctan\left(\dfrac{a}{s}\right)$
47. $\dfrac{\sin at \cos bt}{t}$	$\dfrac{1}{2}\arctan\dfrac{a+b}{s} + \dfrac{1}{2}\arctan\dfrac{a-b}{s}$
48. $\dfrac{1}{\sqrt{\pi t}} e^{-a^2/4t}$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}}$
49. $\dfrac{a}{2\sqrt{\pi t^3}} e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
50. $\operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s}$
51. $2\sqrt{\dfrac{t}{\pi}}\, e^{-a^2/4t} - a \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{s\sqrt{s}}$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
52. $e^{ab}e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{e^{-a\sqrt{s}}}{\sqrt{s}(\sqrt{s} + b)}$
53. $-e^{ab}e^{b^2 t} \operatorname{erfc}\left(b\sqrt{t} + \dfrac{a}{2\sqrt{t}}\right) + \operatorname{erfc}\left(\dfrac{a}{2\sqrt{t}}\right)$	$\dfrac{be^{-a\sqrt{s}}}{s(\sqrt{s} + b)}$
54. $e^{at} f(t)$	$F(s - a)$
55. $\mathscr{U}(t - a)$	$\dfrac{e^{-as}}{s}$
56. $f(t - a)\mathscr{U}(t - a)$	$e^{-as}F(s)$
57. $g(t)\mathscr{U}(t - a)$	$e^{-as}\mathscr{L}\{g(t + a)\}$
58. $f^{(n)}(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
59. $t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
60. $\displaystyle\int_0^t f(\tau) g(t - \tau)\, d\tau$	$F(s)G(s)$
61. $\delta(t)$	1
62. $\delta(t - t_0)$	e^{-st_0}

A First Course in Differential Equations with Modeling Applications

Twelfth Edition

Dennis G. Zill

Loyola Marymount University

Metric Version

prepared by Aly El-Iraki

Professor Emeritus, Alexandria University, Egypt

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Content Project Manager, International: Jyotsna Ojha

Sr Product Director, Portfolio Product Management: Mark Santee

Portfolio Product Director: Rita Lombard

Sr Portfolio Product Manager: Jay Campbell

Product Assistant: Emily Smith

Learning Designer: Laura Gallus

Content Manager: Kelly Aull, Lynh Pham

Digital Project Manager: John Smigielski

Manufacturing Buyer: Elaine Bevan

Sr Content Acquisition Analyst: Ashley Maynard

Production Service: MPS Limited

Designer: Varpu Lauchlan

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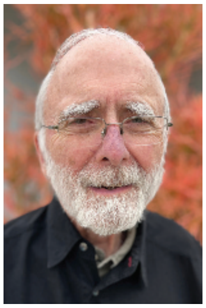
Preface for this Metric Edition



Jacqueline Dewar

Dennis G. Zill
Los Angeles, CA

This International Metric Version differs from the US version of *A First Course in Differential Equations with Modeling Applications*, Twelfth Edition, as follows:

The units of measurement used in most of the examples and exercises have been converted from U.S. Customary Systems (USCS) of units (also referred to as English, or Imperial units) to Metric units.

This Metric Version includes conversion tables to reference as you work through the related applications and exercises.

To The Student

Authors of books live with the hope that someone actually *reads* them. Contrary to what you might believe, almost everything in a typical college-level mathematics text is written for you and not the instructor. True, the topics covered in the text are chosen to appeal to instructors because they make the decision on whether to use it in their classes, but everything written in it is aimed directly at you, the student. So I want to encourage you—no, actually I want to *tell* you—to read this textbook! But do not read this text as you would a novel; you should not read it fast and you should not skip anything. Think of it as a *work*book. By this I mean that mathematics should always be read with pencil and paper at the ready because, most likely, you will have to *work* your way through the examples and the discussion. Before attempting any problems in the section exercise sets, work through *all* the examples in that section. The examples are constructed to illustrate what I consider the most important aspects of the section, and therefore, reflect the procedures necessary to work most of the problems. When reading an example, copy it down on a piece of paper and do not look at the solution in the book. Try working it, then compare your results against the solution given, and, if necessary resolve any differences. I have tried to include most of the important steps in each example, but if something is not clear you should always try—and here is where the pencil and paper come in again—to fill in the details or missing steps. This may not be easy, but it is part of the learning process. The accumulation of facts followed by the slow assimilation of understanding simply cannot be achieved without a struggle.

Specifically for you, a *Student Solutions Manual* (*SSM*) is available as an optional supplement. In addition to containing solutions of selected problems from the exercises sets, the *SSM* contains hints for solving problems, extra examples, and a review of those areas of algebra and calculus that I feel are particularly important to the successful study of differential equations. Bear in mind you do not have to purchase the *SSM*; you can review the appropriate mathematics from your old precalculus or calculus texts.

In conclusion, I wish you good luck and success. I hope you enjoy the text and the course you are about to embark on—as an undergraduate math major it was one of my favorites because I liked mathematics that connected with the physical world. If you have any comments, or if you find any errors as you read/work your way through the text, or if you come up with a good idea for improving either it or the *SSM*, please feel free to contact me through Cengage Learning.

To The Instructor

In case you are examining this text for the first time, *A First Course in Differential Equations with Modeling Applications*, Twelfth Edition, is intended for a one-semester or one-quarter course in ordinary differential equations. The longer version of the text, *Differential Equations with Boundary-Value Problems,* Tenth Edition, can be used for either a one- or two-semester course that covers ordinary and partial differential equations. This text contains six additional chapters. For a one-semester course, it is assumed that the students have successfully completed at least two semesters of calculus. Since you are reading this, undoubtedly you have already examined the table of contents for the topics that are covered. You will not find a "suggested syllabus" in this preface; I will not pretend to be so wise as to tell other teachers what to teach. I feel that there is plenty of material here to choose from and to form a course to your liking. The text strikes a reasonable balance between the analytical, qualitative, and quantitative approaches to the study of differential equations. As far as my "underlying philosophy" goes, it is this: An undergraduate text should be written with the students' understanding kept firmly in mind, which means to me that the material should be presented in a straightforward, readable, and helpful manner, while keeping the level of theory consistent with the notion of a "first course."

For those who are familiar with the previous editions, I would like to mention a few improvements made in this edition. Many exercise sets have been updated by the addition of new problems. Some of these problems involve new and, I think, interesting mathematical models. Additional examples, figures, and remarks have been added to many sections. Throughout the text I have given a greater emphasis to the concepts of piecewise-linear differential equations and solutions that involve nonelementary integrals. Finally, the table of Laplace transforms in Appendix C has been expanded.

Student Resources

- ***Student Solutions Manual* (*SSM*),** prepared by Roberto Martinez (ISBN 979-8-214-03824-7, accompanies *A First Course in Differential Equations with Modeling Applications,* Twelfth Edition, and ISBN 978-0-357-76058-1 accompanies *Differential Equations with Boundary-Value Problems*, Tenth Edition) provides important review material from algebra and calculus, the solution of every third problem in each exercise set (with the exception of the Discussion Problems and Computer Lab Assignments), relevant command syntax for the computer algebra systems *Mathematica* and *Maple*, and lists of important concepts, as well as helpful hints on how to start certain problems.
- **WebAssign for *A First Course in Differential Equations with Modeling Applications***, Twelfth Edition. WebAssign provides you with the tools you need to be successful in differential equations. Course materials and resources have been specially customized for you by your instructor, giving you an array of study tools to get a true understanding of course concepts and achieve better grades.

Instructor Resources

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Chapter

1

Introduction to Differential Equations

1.1 Definitions and Terminology
1.2 Initial-Value Problems
1.3 Differential Equations as Mathematical Models
Chapter 1 In Review

The words *differential* and *equations* suggest solving some kind of equation that contains derivatives y', y'', ... Analogous to a course in algebra, in which a good amount of time is spent solving equations such as $x^2 + 5x + 4 = 0$ for the unknown number x, in this course *one* of our tasks will be to solve differential equations such as $y'' + 2y' + y = 0$ for an unknown function $y = \phi(x)$. As the course unfolds, you will see there is more to the study of differential equations than just mastering methods that mathematicians over past centuries devised to solve them. But first things first. In order to read, study, and be conversant in a specialized subject you have to learn some of the terminology of that discipline. This is the thrust of the first two sections of this chapter. In the last section we briefly examine the link between differential equations and the real world.

1.1 Definitions and Terminology

Introduction The derivative dy/dx of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. The exponential function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$ and by the Chain Rule its first derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ on the right-hand side of the last equation by the symbol y, the derivative becomes

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

Now imagine that a friend of yours simply hands you equation (1)—you have no idea how it was constructed—and asks, *What is the function represented by the symbol y?* You are now face to face with one of the basic problems in this course:

How do you solve an equation such as (1) for the function $y = \phi(x)$?

A Definition The equation that we made up in (1) is called a **differential equation.** Before proceeding any further, let us consider a more precise definition of this concept.

Definition 1.1.1 Differential Equation

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE).**

To talk about them, we shall classify differential equations according to **type, order,** and **linearity.**

Classification by Type If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a *single* independent variable, it is said to be an **ordinary differential equation (ODE).** An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a **partial differential equation (PDE).** Our first example illustrates several of each type of differential equation.

Example 1 Types of Differential Equations

(a) The equations

an ODE can contain more than one unknown function

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \tag{2}$$

are examples of ordinary differential equations.

(b) The following equations are partial differential equations:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3}$$

*Except for this introductory section, only ordinary differential equations are considered in *A First Course in Differential Equations with Modeling Applications*, Twelfth Edition. In that text the word *equation* and the abbreviation DE refer only to ODEs. Partial differential equations or PDEs are considered in the expanded volume *Differential Equations with Boundary-Value Problems*, Tenth Edition.

Notice in the third equation that there are two unknown functions and two independent variables in the PDE. This means u and v must be functions of *two or more* independent variables. ■

Notation The German polymath **Gottfried Wilhelm Leibniz** (1646–1716) along with the English mathematician, physicist, and theologian **Sir Isaac Newton** (1643–1727) are considered to be the co-inventors of calculus. But much of the notation used today in mathematics is due to Leibniz and the Italian mathematician and astronomer **Joseph-Louis Lagrange** (1736–1813). See Figures 1.1.1–1.1.3. For example, the familiar symbols

$$\frac{dy}{dx} \qquad \text{and} \qquad f'(x)$$

are due, respectively, to Leibniz and Lagrange. Throughout this text ordinary derivatives will be written using either the Leibniz notation

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots$$

or the Lagrange prime notation

$$y', y'', y''', \ldots .$$

By using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 5y = e^x$ and $y'' - y' + 6y = 0$. Actually the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''. In general, the nth derivative of y is written d^ny/dx^n or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the differential equation

$$\frac{d^2x}{dt^2} + 16x = 0$$

it is immediately seen that the symbol x now represents a dependent variable whereas the independent variable is t. You should also be aware that in physical sciences and engineering, **Newton's dot notation** (derogatively referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time t. Thus, the differential equation

$$\frac{d^2s}{dt^2} = -32 \qquad \text{becomes} \qquad \ddot{s} = -32.$$

Partial derivatives such as $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are often denoted by a **subscript notation** indicating the independent variables. For example, the first and second equations in (3) can be written, in turn,

$$u_{xx} + u_{yy} = 0 \qquad \text{and} \qquad u_{xx} = u_{tt} - 2u_t.$$

Classification by Order The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

$$\overbrace{\frac{d^2y}{dx^2}}^{\text{second order}} + 5\left(\overbrace{\frac{dy}{dx}}^{\text{first order}}\right)^3 - 4y = e^x$$

is a second-order ordinary differential equation. In Example 1, the first and third equations in (2) are first-order ODEs, whereas in (3) the first two equations are second-order PDEs. A first-order ordinary differential equation is sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

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Figure 1.1.1 Gottfried Wilhelm Leibniz

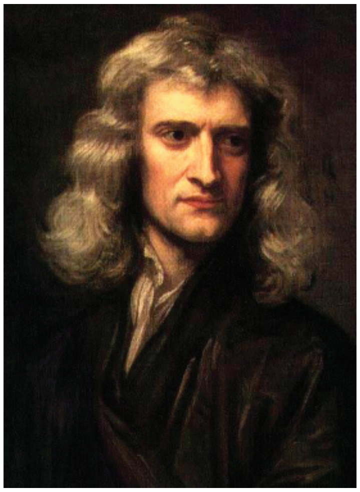

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Figure 1.1.2 Sir Isaac Newton

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Figure 1.1.3 Joseph-Louis Lagrange

Example 2 Differential Form of a First-Order ODE

If we assume that y is the dependent variable in a first-order ODE, then recall from calculus that the differential dy is defined to be $dy = y'\,dx$.

(a) By dividing by the differential dx an alternative form of the equation $(y - x)\,dx + 4x\,dy = 0$ is given by

$$y - x + 4x\frac{dy}{dx} = 0 \quad \text{or equivalently} \quad 4x\frac{dy}{dx} + y = x.$$

(b) By multiplying the differential equation

$$6xy\frac{dy}{dx} + x^2 + y^2 = 0$$

by dx we see that the equation has the alternative differential form

$$(x^2 + y^2)\,dx + 6xy\,dy = 0.$$

In symbols we can express an nth-order ordinary differential equation in one dependent variable by the general form

$$F(x,\ y,\ y',\ldots, y^{(n)}) = 0, \tag{4}$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where f is a real-valued continuous function, is referred to as the **normal form** of (4). Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations.

Example 3 Normal Form of an ODE

(a) By solving for the derivative dy/dx the normal form of the first-order differential equation

$$4x\frac{dy}{dx} + y = x \quad \text{is} \quad \frac{dy}{dx} = \frac{x - y}{4x}.$$

(b) By solving for the derivative y'' the normal form of the second-order differential equation

$$y'' - y' + 6y = 0 \quad \text{is} \quad y'' = y' - 6y.$$

Classification by Linearity An nth-order ordinary differential equation (4) is said to be **linear** if F is linear in $y, y', \ldots, y^{(n)}$. This means that an nth-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{6}$$

Two important special cases of (6) are linear first-order ($n = 1$) and linear second-order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad \text{and} \qquad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{7}$$

In the additive combination on the left-hand side of equation (6) we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation.

Example 4 Linear and Nonlinear ODEs

(a) The equations

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y + y = 0, \quad x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated in part (a) of Example 2 that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

$$\underbrace{(1 - y)}_{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}}y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underbrace{\sin y}_{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}} = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + \underbrace{y^2}_{\substack{\text{nonlinear term:}\\ \text{power not 1}}} = 0$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively.

(c) By using the quadratic formula the nonlinear first-order differential equation $(y')^2 + 2xy' - y = 0$ is equivalent to two nonlinear first-order equations in normal form

$$y' = -x + \sqrt{x^2 + y} \qquad \text{and} \qquad y' = -x - \sqrt{x^2 + y}.$$

Solutions As was stated on page 2, one of the goals in this course is to solve, or find solutions of, differential equations. In the next definition we consider the concept of a solution of an ordinary differential equation.

Definition 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an nth-order ordinary differential equation (4) is a function ϕ that possesses at least n derivatives and for which

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

We say that ϕ *satisfies* the differential equation on I. For our purposes we shall also assume that a solution ϕ is a real-valued function. In our introductory discussion we saw that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally, it will be convenient to denote a solution by the alternative symbol $y(x)$.

Interval of Definition You cannot think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval I in Definition 1.1.2 is variously called the **interval of definition,** the **interval of existence,** the **interval of validity,** or the **domain of the solution** and can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

Example 5 Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}$; $y = \frac{1}{16}x^4$ **(b)** $y'' - 2y' + y = 0$; $y = xe^x$

Solution One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x in the interval.

(a) From

$$\textit{left-hand side:} \quad \frac{dy}{dx} = \frac{1}{16}(4 \cdot x^3) = \frac{1}{4}x^3,$$

$$\textit{right-hand side:} \quad xy^{1/2} = x \cdot \left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3,$$

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

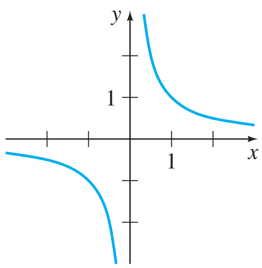
(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number x,

$$\textit{left-hand side:} \quad y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$$

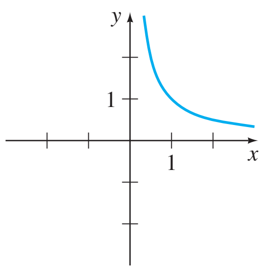
$$\textit{right-hand side:} \quad 0. \qquad \blacksquare$$

Note, too, that each differential equation in Example 5 possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution.**

Solution Curve The graph of a solution ϕ of an ODE is called a **solution curve.** Since ϕ is a differentiable function, it is continuous on its interval I of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. Put another way, the domain of the function ϕ need not be the same as the interval I of definition (or domain) of the solution ϕ. Example 6 illustrates the difference.



(a) function $y = 1/x$, $x \neq 0$



(b) solution $y = 1/x$, $(0, \infty)$

Figure 1.1.4 In Example 6 the function $y = 1/x$ is not the same as the solution $y = 1/x$

Example 6 Function versus Solution

(a) The domain of $y = 1/x$, considered simply as a *function*, is the set of all real numbers x except 0. When we graph $y = 1/x$, we plot points in the xy-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in Figure 1.1.4(a). The function $y = 1/x$ is not differentiable at $x = 0$, since the y-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

(b) Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$. (Verify.) But when we say that $y = 1/x$ is a *solution* of this DE, we mean that it is a function defined on an interval I on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval that does not contain 0, such as $(-3, -1)$, $\left(\frac{1}{2}, 10\right)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ for $-3 < x < -1$ and $\frac{1}{2} < x < 10$ are simply segments, or pieces, of the solution curves defined by $y = 1/x$ for $-\infty < x < 0$ and $0 < x < \infty$, respectively, it makes sense to take the interval I to be as large as possible. Thus we take I to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on $(0, \infty)$ is shown in Figure 1.1.4(b). ■

Explicit and Implicit Solutions You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution.** For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution ϕ implicitly.

Definition 1.1.3 Implicit Solution of an ODE

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval I, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

It is beyond the scope of this course to investigate the conditions under which a relation $G(x, y) = 0$ defines a differentiable function ϕ. So we shall assume that if the formal implementation of a method of solution leads to a relation $G(x, y) = 0$, then there exists at least one function ϕ that satisfies both the relation (that is, $G(x, \phi(x)) = 0$) and the differential equation on an interval I. If the implicit solution $G(x, y) = 0$ is fairly simple, we may be able to solve for y in terms of x and obtain one or more explicit solutions. See (*iv*) in the *Remarks.*

Example 7 Verification of an Implicit Solution

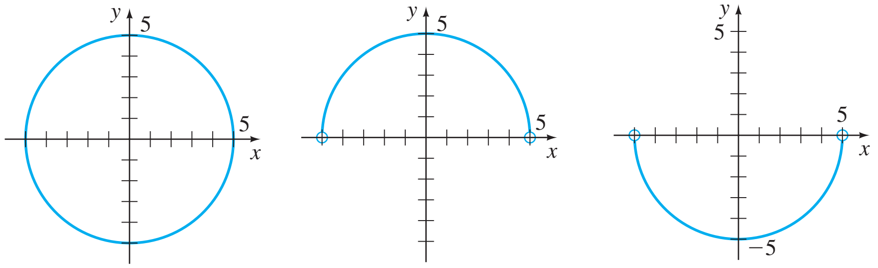
The relation $x^2 + y^2 = 25$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{8}$$

on the open interval $(-5, 5)$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0. \tag{9}$$

Solving the last equation in (9) for the symbol dy/dx gives (8). Moreover, solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $(-5, 5)$. The solution curves given in Figures 1.1.5(b) and 1.1.5(c) are segments of the graph of the implicit solution in Figure 1.1.5(a).



(a) implicit solution $x^2 + y^2 = 25$

(b) explicit solution $y_1 = \sqrt{25 - x^2},\ -5 < x < 5$

(c) explicit solution $y_2 = -\sqrt{25 - x^2},\ -5 < x < 5$

Figure 1.1.5 An implicit solution and two explicit solutions of (8) in Example 7

Because the distinction between an explicit solution and an implicit solution should be intuitively clear, we will not belabor the issue by always saying, "Here is an explicit (implicit) solution."

Families of Solutions The study of differential equations is similar to that of integral calculus. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration. Analogously, we shall see in Chapter 2 that when solving a first-order differential equation $F(x, y, y') = 0$ we *usually* obtain a solution containing a single constant or parameter c. A solution of $F(x, y, y') = 0$ containing a constant c is a set of solutions $G(x, y, c) = 0$ called a **one-parameter family of solutions.** When solving an nth-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameter(s). A solution of a differential equation that is free of parameters is called a **particular solution.**

The parameters in a family of solutions such as $G(x, y, c_1, c_2, \ldots, c_n) = 0$ are *arbitrary* up to a point. For example, proceeding as in (9) a relation $x^2 + y^2 = c$ formally satisfies (8) for any constant c. However, it is understood that the relation should always make sense in the real number system; thus, if $c = -25$ we cannot say that $x^2 + y^2 = -25$ is an implicit solution of the differential equation.

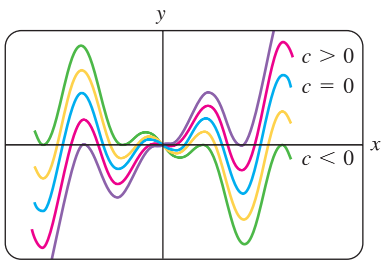


Figure 1.1.6 Some solutions of DE in part (a) of Example 8

Example 8 Particular Solutions

(a) For all real values of c, the one-parameter family $y = cx - x \cos x$ is an explicit solution of the linear first-order equation

$$xy' - y = x^2 \sin x$$

on the interval $(-\infty, \infty)$. (Verify.) Figure 1.1.6 shows the graphs of some particular solutions in this family for various choices of c. The solution $y = -x \cos x$, the blue graph in the figure, is a particular solution corresponding to $c = 0$.

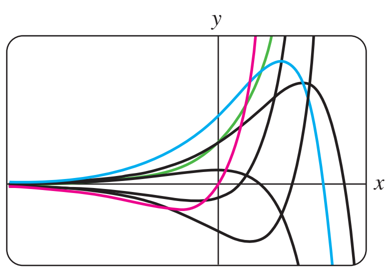


Figure 1.1.7 Some solutions of DE in part (b) of Example 8

(b) The two-parameter family $y = c_1 e^x + c_2 x e^x$ is an explicit solution of the linear second-order equation

$$y'' - 2y' + y = 0$$

in part (b) of Example 5. (Verify.) In Figure 1.1.7 we have shown seven of the "double infinity" of solutions in the family. The solution curves in red, green, and blue are the graphs of the particular solutions $y = 5xe^x$ ($c_1 = 0$, $c_2 = 5$), $y = 3e^x$ ($c_1 = 3$, $c_2 = 0$), and $y = 5e^x - 2xe^x$ ($c_1 = 5$, $c_2 = -2$), respectively.