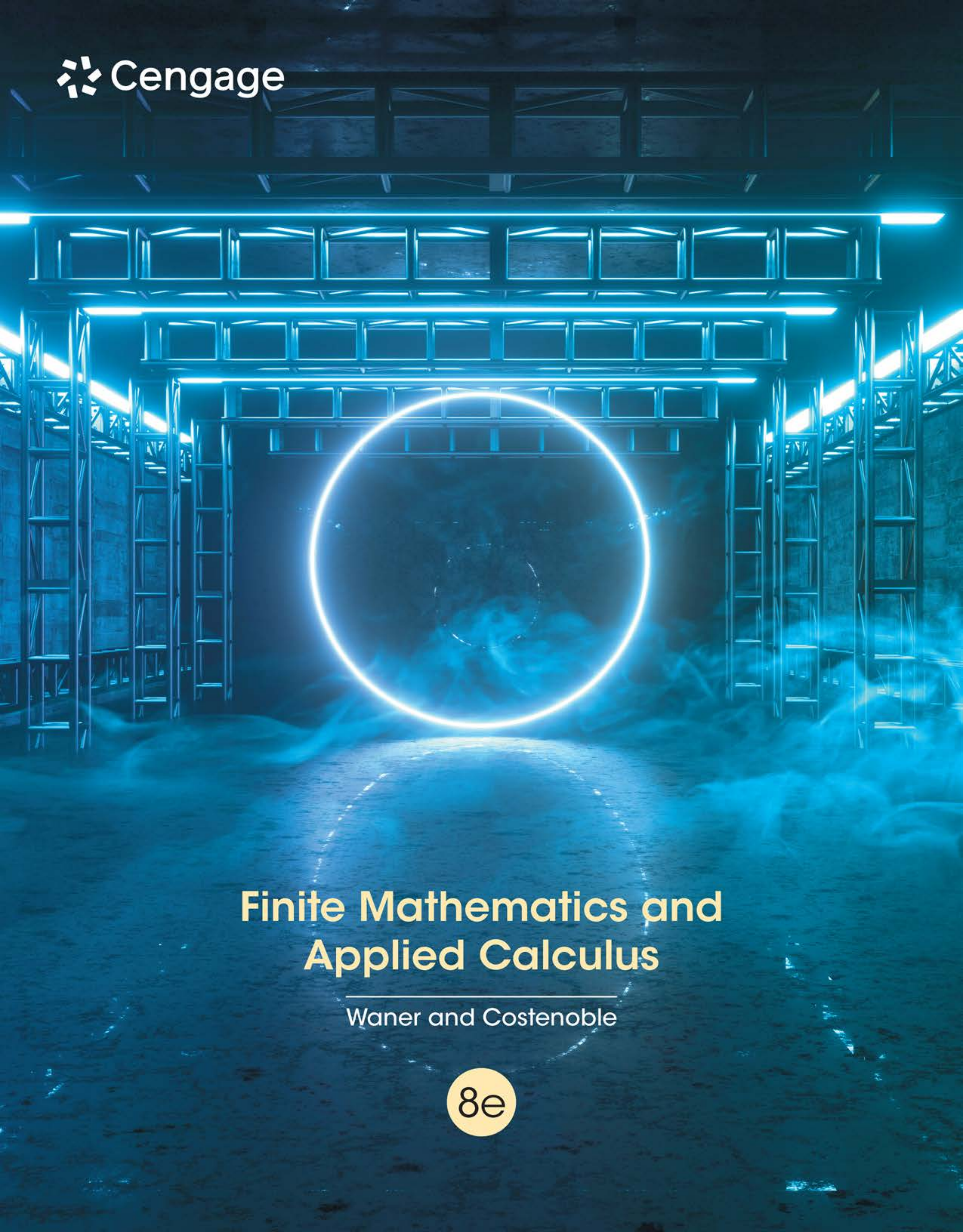

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Finite Mathematics and Applied Calculus

Waner and Costenoble

8e

Finite Mathematics and Applied Calculus

Finite Mathematics and Applied Calculus

EIGHTH EDITION

Stefan Waner
Hofstra University

Steven R. Costenoble
Hofstra University

Cengage

Australia • Brazil • Canada • Mexico • Singapore • United Kingdom • United States

Cengage

***Finite Mathematics and Applied Calculus*, Eighth Edition**
Stefan Waner, Steven R. Costenoble

SVP, Product: Erin Joyner

VP, Product: Thais Alencar

Portfolio Product Director: Mark Santee

Portfolio Product Manager: Jay Campbell

Product Assistant: Amelia Padilla

Learning Designer: Susan Pashos

Content Manager: Shelby Blakey

Digital Project Manager: John Smigielski

Product Marketing Manager: Taylor Shenberger

Content Acquisition Analyst: Nichole Nalenz

Production Service: Lumina Datamatics Ltd.

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BRIEF CONTENTS

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ABOUT THE AUTHORS

Stefan Waner and Steven R. Costenoble both received their Ph.D. from the University of Chicago, having studied several years apart with the same advisor, J. Peter May. Their paths merged when Steven joined Stefan at Hofstra University in 1987; since then, they have coauthored 18 research papers as well as a research-level monograph in algebraic topology. By the early 1990s they had become dissatisfied with many of the Finite Mathematics and Applied Calculus textbooks. They wanted textbooks that were more readable and relevant to students' interests, containing examples and exercises that were interesting, and reflected the interactive approaches and techniques they found worked well with their own students. It therefore seemed natural to extend their research collaboration to a joint textbook writing project that expressed these ideals. To this day, they continue to work together on their textbook projects, their research in algebraic topology, and their teaching.

PREFACE

Finite Mathematics and Applied Calculus is intended for a one- or two-term course for students majoring in business, the social sciences, or the liberal arts. The eighth edition is designed to address two challenges: (1) generating enthusiasm and mathematical sophistication in a student audience that may not have optimum preparation and may not be intrinsically motivated by traditional mathematics courses, and (2) providing the structure and flexibility to support instructors in implementing a wide variety of pedagogical philosophies and instructional paradigms.

In the eighth edition we meet these challenges through significant enhancements in content and context:

- we focus on **up-to-date real-life applications** that students can relate to and instructors can use to engage their audience,
- we present mathematical concepts **intuitively and thoroughly** with a writing style that is **informal, engaging, and occasionally even humorous**,
- we have **streamlined and enhanced** the thematic structure throughout the book,
- we support **all levels of technology usage.** Our text provides comprehensive support for various technologies, including graphing calculators, spreadsheets, and our powerful online utilities. This is carefully implemented to make this edition work seamlessly in courses that use no technology, that focus on a single form of technology, or that incorporate several technologies.

No previous calculus background is assumed. Although we provide extensive review of precalculus skills and concepts in Chapter 0, students should have completed college algebra or the equivalent as a prerequisite.

Complete Support for Various Course Needs and Learning Environments

In this edition we bring you much more than just a book: In combination with the **WebAssign** online homework platform and ebook as well as the extensive author-developed website at **AppliedMathSite.net**, we can now deliver a fully interactive version of this book including adaptive practice for examples linked to a large and growing knowledge base of support topics.

We integrate resources like gamified interactive tutorials, powerful online tools, and section-by-section videos. These provide instructors with the resources they need in a range of class structures and delivery styles, from traditional on-campus classes (with or without technology support) to various types of hybrid and online classes.

In addition to the trusted homework experience, you'll find an array of tools and resources in WebAssign that support other areas of your course.

- Students can practice **prerequisite** algebra and pre-calculus skills and concepts using WebAssign boot camps, curated practice assignments, as well as the extensive tutorials at **AppliedMathSite.net** (links to which are provided in WebAssign).

- You can assign **curated pre- or post-class assessments** in WebAssign, containing a mix of quantitative and qualitative questions. The assessments ensure students complete any assigned pre-class work and/or review material covered in class.
- WebAssign and **AppliedMathSite.net** offer **additional in-class support**, like alternative worked examples and individual or group activities.
- Students can **study** for tests and quizzes in a self-paced environment using curated review assignments in WebAssign and the extensive topic summaries and review tutorials at **AppliedMathSite.net** (links to which are also provided in WebAssign).
- You can utilize WebAssign as a **quizzing and testing** platform with curated test assignments with built-in academic integrity features.

Our Approach to Pedagogy

Real World Orientation The diversity, breadth and abundance of examples and exercises included in this edition continue to distinguish our book from the others. Many of our examples and exercises, as well as the capstone "Case Study" projects at the end of each chapter, are based on real, referenced, and timely data from business, economics, the life sciences, and the social sciences.

The sheer number of new and updated examples and exercises in the eighth edition is unprecedented in earlier editions, and hence focus more on what students and instructors now see around them. The updated material covers topics ranging from the rapid rise of volatile cryptocurrencies and a sudden unanticipated global pandemic to the network technology students constantly use in their phones and the corporations they will instantly recognize as important in their lives. Historically notable events like the 1990s Dot Com boom, the 2005–2006 real estate bubble and resulting 2008 economic crisis, the 2010 stock market "Flash Crash," the earlier outbreaks of SARS and Ebola in 2003 and 2014, and many more, are also addressed in our examples and exercises throughout the book.

As in earlier editions, we have been careful to strike a pedagogically sound balance between applications based on real data and more traditional "generic" applications. Thus, the density and selection of real data-based applications continues to be tailored to the pedagogical goals and appropriate difficulty level for each section.

Streamlined Thematic Structure and Readability As in previous editions, our writing style is geared to our desire to have students read this book, and to *enjoy* reading this book. Thus, we present the material in a conversational and student-oriented style and have made frequent use of question-and-answer dialogues to encourage the development of the student's mathematical curiosity and intuition. Equally important is the need, for students and instructors alike, for a carefully delineated thematic structure so that students can focus on the central ideas in each topic and refer quickly to appropriate portions of the text, and instructors can more easily organize their presentations.

The eighth edition is a significant revision in this latter regard, and we have streamlined the discussion significantly and enhanced the organization of topics within each section to create a balance between conversational style and a careful focus on the mathematical ideas presented. The topics students will see and the skills they will master in each section appear as a brief marginal *Topics and Skills* list at the beginning of each section. Moreover, through new *Practice This Skill* marginal elements, the examples in the text now point to specific exercises based on those examples, just as many exercises point back to specific examples on which they are based.

We also have reimagined and revised one of our main features, the *Key Concept* pedagogy boxes, which introduce each new concept along with short *Quick Examples* to reinforce and give immediate context to these concepts the moment they are introduced.

Pedagogical Aids Our series has always distinguished itself through innovative approaches to mastering concepts that normally cause difficulties for students and headaches for instructors, and this edition is no exception. For instance,

Key Concept **Linear Function**

A **linear function** is one that can be written in the form

$$f(x) = mx + b \qquad \text{Function form}$$

or

$$y = mx + b \qquad \text{Equation form}$$

where m and b are fixed numbers (the names m and b are traditional*).

QUICK EXAMPLE

$f(x) = 3x - 1$

$y = 3x - 1$

- We present a rewording technique in Chapters 4 and 6 to show how to translate phrases like "there are (at least/at most) three times as many X as Y" directly into equations or inequalities.
- The counting arguments in Chapter 7 are cast in terms of "decision algorithms" that make counting involving permutations and combinations almost mechanical.
- Which limits can and cannot be evaluated by simple substitution is made much less mysterious through a catch-all "Theorem C" in Chapter 10 that allows a student to immediately know when substitution does and does not work.
- Also in Chapter 10, limits at infinity involving polynomials are calculated quickly using a theorem on when to discard lower order terms, avoiding the tedious and repetitive ritual of having to divide top and bottom by the highest power of x. (The proof of the theorem does that once and for all.)
- Verbal forms of the differentiation rules in Chapter 11 help students avoid having to rely on multiple formulas they might not really understand, and "calculation thought experiments" help the student decide which rules of differentiation to apply and the order in which to apply them.
- A powerful tabular method for integration by parts in Chapter 14 transforms what is often an agonizingly complicated topic for students into one that is far simpler and almost mechanical.

Topics and Skills

TOPICS

- Resource allocation
- Flow analysis
- Transportation analysis

SKILLS

- Analyzing and solving real world problems that lead to systems of linear equations (all Examples)

Practice This Skill

Try Exercises 1–4.
Online: AppliedMathSite.net
Practice and visualize → Solving a blending problem

Rigor Mathematical rigor need not be antithetical to the kind of focus on concepts and applications that are hallmarks of this book. We have worked hard to ensure that we are always mathematically honest without being unnecessarily formal. Sometimes we do this through the question-and-answer dialogs and sometimes through the "Before we go on…" discussions that follow examples, but always in a manner designed to provoke the interest of the student.

Five Elements of Mathematical Pedagogy to Address Different Learning Styles The "Rule of Four" is a common theme in many mathematics texts. In implementing this approach, we discuss many of the central concepts numerically, graphically, and algebraically, and clearly delineate these distinctions. The fourth element, verbal communication of mathematical concepts, is emphasized through our discussions on translating English sentences into mathematical statements, and in our numerous Communication and Reasoning exercises at the end of each section.

The fifth element, interactivity, is taken to a new level in the eighth edition with fully embedded interactive elements and randomizable *Your Turn* versions of all key examples in the eBook in WebAssign. In the printed version, these elements are pointed to via marginal notes that allow them to be easily located at the **AppliedMathSite.net** website with a couple of clicks. Each *Your Turn* example is equipped with an adaptive practice session—often with variable levels of difficulty—tied to a large and expanding knowledge base that can guide a student experiencing difficulties with a particular example all the way back to the practice of the requisite basic algebra skills. Instructors can also assign the *Your Turn* elements to count for credit, delivered either as a regular assignment or in a gamified form analogous to the online game tutorials.

In addition, **AppliedMathSite.net** offers interactive tutorials in the form of games, interactive chapter summaries, and chapter review exercises, and online utilities that automate a variety of tasks, from graphing and linear programming to regression and matrix algebra.

Exercise Sets Comprehensive and wide-ranging exercise sets have been a hallmark of our series, and with each successive edition we have continued to refine and update them to ensure that they continue to provide material that is timely and relevant, while remaining varied enough to challenge students at almost every level of preparation, including everything from straightforward drill exercises to interesting and rather challenging applications and conceptual exercises. The exercise sets are carefully graded to move from basic exercises and exercises that are similar to examples in the text to more interesting and advanced ones, marked as "more advanced" (▼) for easy reference. There are also several much more difficult exercises, designated as "challenging" (◆). We have also included, in virtually every section of every chapter, numerous interesting applications based on real data, and in these applications the instructor will notice the large number of new and updated versions in the eighth edition, reflecting the many recent and sometimes dramatic changes in the world around us. Equally important are the Communication and Reasoning exercises that help students articulate mathematical concepts and recognize common errors, and exercises indicated for the use of technology (T).

Some of the scenarios used in application examples and exercises are revisited several times throughout the book. Thus, for instance, students will find themselves using a variety of techniques, from solving systems of equations to linear programming, or graphing through the use of derivatives and elasticity, to analyze the same application. Reusing scenarios and important functions provides unifying threads and shows students the complex texture of real-life problems.

New To This Edition

In addition to the new *Key Concept* structure throughout the text and *Topics and Skills* boxes that begin each section, we have made the following changes within individual chapters.

- **Chapter 0: Precalculus Review** Recognizing that exponents and radicals are often areas of weakness in student preparation, we have split the original Section 0.2 on that subject into two sections: the first to introduce and discuss the various identities of exponents and radicals, and the second devoted to the use of these identities to represent algebraic expressions involving them in power form (used in calculus), positive exponent form (often used for solving equations), and simplest radical form (to express answers involving radicals in the lowest possible terms). We have also extended the discussion of logarithms (Section 0.9) by including material on change-of-base formulas and related properties of logarithms that was originally in the section on exponential functions in Chapter 2.
- **Chapter 1: Functions and Applications** In our revision of this important introductory chapter, we have streamlined much of the material by reducing the size and complexity of data sets used in the examples and exercises, removing some unnecessary discussion, and postponing the discussion of some topics (like carbon dating) that are addressed more extensively in Chapter 2. In particular, Section 1.2 on mathematical models is now significantly more concise and manageable.

- **Chapter 2: Nonlinear Functions and Models** This chapter has been significantly revised. We have split the original long section on exponential functions into two: the first on exponential functions written in the form Ab^x and the second on the number e, exponential functions in the form Ae^{kx}, and exponential growth and decay. The originally shorter sections on logarithmic and logistic functions and models are now combined into a single section that, among other things, can now compare the two kinds of function in modeling long-term trends. The basic material on change-of-base formulas and related properties of logarithms has been moved to Chapter 0 where it properly belongs, and the material on doubling time, half-life and solving for exponents, which is not directly about logarithmic functions, is now in Sections 2 and 3 where it fits more comfortably. In their place, we have included more discussion of real-life logarithmic models that had been largely left to the exercises in the seventh edition. The Chapter 2 Case Study is also new: a modified and updated form of the original Chapter 11 Case Study now discussing recent trends in college enrollments.

- **Chapter 3: Mathematics of Finance** In general, the discussions in this chapter have been narrowed to discuss and use fewer variations on each formula than we used in the seventh edition. In the sections on simple and compound interest, we now give only one version of each formula, based on time t measured in any convenient units (not necessarily years) and the interest rate r being the rate per unit of time, allowing for a simpler presentation than in previous editions. In Section 3 on annuities, we simplify further by insisting that t be measured in compounding periods, so that the formulas involve only t and r, which is now the interest rate per compounding period. This change is consistent with the formulas used in the preceding sections and also with common usage.

- **Chapter 5: Matrix Algebra** The Chapter 5 Case Study on social networks is completely new and demonstrates how the concept of an eigenvector—although normally beyond the scope of a book like this—introduces itself naturally as a measure of participants' degrees of influence in the network.

- **Chapter 10: Introduction to the Derivative** The first three sections in the introductory calculus chapter have been reorganized to present limits and continuity in a more logically appealing order: limits numerically, graphically and algebraically, and then continuity. In the section on limits algebraically, we introduce "the easy way to calculate limits algebraically" ('Theorem C:' limits of closed form functions at points in their domains) and then what to do at singular points. This presentation sets the stage of a more streamlined discussion of continuity that is no longer burdened by technicalities on calculating limits.

- **Chapter 11: Techniques of Differentiation** The Chapter 11 Case Study is completely new and applies logistic functions and their derivatives to obtain a surprisingly accurate model of the first three waves of the COVID-19 epidemic in the U.S.

- **Chapter 12: Further Applications of the Derivative** Our discussion of maxima and minima now begins with the more intuitive and simpler notion of absolute extrema, followed by relative extrema. Among the latter we continue to include, as candidates, endpoints and singular points of the derivatives, so that extrema that are not absolute are automatically relative. In the section on analysis of graphs, we now present two distinct approaches: a "by-hand" curve sketching approach that does not rely on graphing technology, and a "using technology" approach that uses graphs of the function and its derivatives in different viewing windows to accurately identify and locate the interesting features.

Continuing Features

- **Quick Examples** Unique to our series, these short straightforward examples are included immediately after the introduction of new concepts or mathematical terms, and thus solidify the new ideas for a student the moment they are introduced. So, by the time students reach the more extensive "regular" examples, they are already familiar with concrete instances of these ideas and can focus on their application.

Key Concept **Functions with Equal Limits**

If $f(x) = g(x)$ for all x except possibly $x = a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

QUICK EXAMPLES

8. $f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x - 1)(x + 1)}{x - 1} = x + 1$ for all x except $x = 1$

Therefore,

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1}(x + 1) = 1 + 1 = 2.$$

- **Case Studies** The end-of-chapter "Case Studies" are extended applications that use and illustrate the central ideas of each chapter, focusing on the development of mathematical models appropriate to the topics. These applications are ideal for assignment as projects, and to this end we include groups of exercises at the end of each.


iStock.com/Martin-dm

Case Study

Tracking the COVID-19 Epidemic in the U.S.

It is mid-March in 2021, and the U.S. has just emerged from three waves of COVID-19 and is hoping that the large-scale vaccination program now underway can prevent a fourth wave. As a consultant to the Centers for Disease Control and Prevention (CDC), you have been asked to produce a mathematical model that tracks the daily cases as well as accumulated cases separately for each of the three waves during the entire pandemic so far. Moreover, recent figures for daily cases show a recent uptick that has some officials in the CDC worried. Is that uptick some kind of mathematical consequence of a three-wave model of the epidemic, or does it herald a possible new wave that the CDC needs to prepare for?

- **Before We Go On** Most examples conclude with supplementary discussions, which may include a check on the answer, a discussion of the feasibility and significance of a solution, or an in-depth look at what the solution means.

Before we go on . . . The simplest way to find the interest earned on the T-bill is by subtraction:

$$INT = FV - PV = 10{,}000 - 9{,}819.81 = \$180.19.$$

So, after 6 months we received back \$10,000: the sum of our original investment and \$180.19 in interest.

- **Question and Answer Dialogue** We frequently use informal question-and-answer dialogues that anticipate the kinds of questions that may occur to the student and also guide the student through the development of new concepts.

FAQ: When to Use an Exponential Model

Q: Given a set of data points that appear to be curving upward, how can I tell whether to use a quadratic model or an exponential model?

A: Here are some things to look for:

- Do the data values appear to double at regular intervals? (For example, do the values approximately double every 5 units?) If so, then an exponential model is appropriate. If it takes longer and longer to double, then a quadratic model may be more appropriate.
- Do the values first decrease to a low point and then increase? If so, then a quadratic model is more appropriate.

It is also helpful to use technology to graph both the regression quadratic and exponential curves and to visually inspect the graphs to determine which gives the closest fit to the data. ■

- **Marginal Technology Notes** Our brief marginal technology notes outlining the use of graphing calculators, spreadsheets, and website technologies in the accompanying examples have been shortened, but those with some experience using these technologies will likely not require more detail. When such detail is desired or indicated, the reader can refer to the extensive discussions in the end-of-chapter Technology Guides.

- **End-of-Chapter Technology Guides** We continue to include detailed TI-83/84 Plus and Spreadsheet Guides at the end of each chapter. These Guides are referenced liberally in marginal technology notes at appropriate points in the chapter, so instructors and students can easily use this material or not, as they prefer. Groups of exercises for which the use of technology is suggested or required appear throughout the exercise sets.

T Using Technology

TI-83/84 PLUS

[2nd] [CATALOG]

`DiagnosticOn`

Then [STAT] CALC option #4:

`LinReg?(ax+b)`

[More details in the Technology Guide.]

SPREADSHEET

Add a trendline and select the option to "Display R-squared value on chart."

[More details and other alternatives in the Technology Guide.]

WWW.APPLIEDMATHSITE.NET

The following two utilities will show regression lines and also r^2 (link to either from Math Tools for Chapter 1):

Simple regression utility

function evaluator and grapher

TI-83/84 Technology Guide

Section 3.2

Example 1 (page 203) In October 2020, **Radius Bank** was paying 0.15% annual interest on savings accounts with balances of \$2,500 to \$24,999.99. If the interest is compounded quarterly, find the future value of a \$2,500 deposit after 6 years. What is the total interest paid over the time of the investment?

SOLUTION

We could calculate the future value using the TI-83/8[…] by entering

`2500(1+0.0015/4)^(4*6)`

- *PMT* is not used in this example (it will be used in the next section) and should be 0.
- *FV* is the future value, which we shall compute in a moment; it doesn't matter what you enter now.
- *P/Y* and *C/Y* stand for payments per year and compounding periods per year, respectively, but we can use any unit of time we like, not only years. They should both be set

Spreadsheet Technology Guide

Section 3.2

Example 1 (page 203) In October 2020, **Radius Bank** was paying 0.15% annual interest on savings accounts with balances of \$2,500 to \$24,999.99. If the interest is compounded quarterly, find the future value of a \$2,500 deposit after 6 years. What is the total interest paid over the time of the investment?

SOLUTION

You can either compute compound interest directly or use financial functions built into your spreadsheet. The following worksheet has more than we need for this example but will be useful for other examples in this and the next section. Although this spreadsheet is labeled assuming time is measured in years, you can use it with time measured in

	A	B	C	D
1		Entered	Calculated	
2	Rate	0.15%		
3	Years	6		
4	Payment	\$0.00		
5	Present Value	(\$2,500.00)		
6	Future Value		\$2,522.60	
7	Periods per year	4		

Note that we have formatted the cells B4:C6 as currency with two decimal places. If you change the values in column B, the future value in column C will be automatically recalculated.

Example 2 (page 204) *Megabucks Corporation* is issuing 10-year bonds. How much would you pay for bonds with a maturity value of \$10,000 if you wished to get a return of 6.5% compounded annually?

- **Communication and Reasoning Exercises for Writing and Discussion** These are exercises designed to broaden the student's grasp of the mathematical concepts and develop modeling skills. They include exercises in which the student is asked to provide his or her own examples to illustrate a point or design an application with a given solution. They also include "fill in the blank" type exercises, exercises that invite discussion and debate, and—perhaps most importantly—exercises in which the student must identify and correct common errors. These exercises often have no single correct answer.

Communication and Reasoning Exercises

93. Which of the following three functions will be largest for large values of x?
(A) $f(x) = x^2$
(B) $r(x) = 2^x$
(C) $h(x) = x^{10}$

94. Which of the following three functions will be smallest for large values of x?
(A) $f(x) = x^{-2}$
(B) $r(x) = 2^{-x}$
(C) $h(x) = x^{-10}$

Cengage's Online Learning Platform: WebAssign

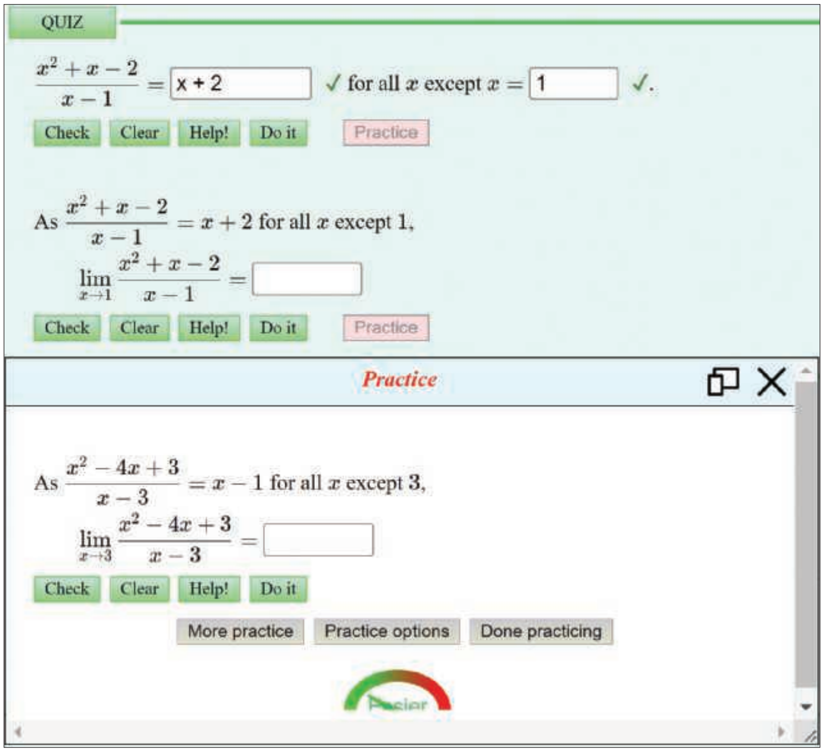
Built by educators, WebAssign provides flexible settings at every step to customize your course with online activities and secure testing to meet learners' unique needs. Students get everything in one place, including rich content and study resources designed to fuel deeper understanding, plus access to a dynamic, interactive ebook. Proven to help hone problem-solving skills, WebAssign helps you help learners in any course format.

The WebAssign course for this title has more than 4,000 available questions, course packs containing curated exercise sets, and integrated resources from **AppliedMathSite.net**. To learn more, visit https://www.cengage.com/webassign.

The Authors' Website: AppliedMathSite.net

The authors' website, accessible through **www.AppliedMathSite.net**, has been evolving for close to three decades with growing recognition in many parts of the world. Students, raised in an environment in which the Web suffuses both work and play, can use their browsers to engage with the material in an active way. The following features of **AppliedMathSite.net** are fully integrated with the text and can be used as a personalized study resource:

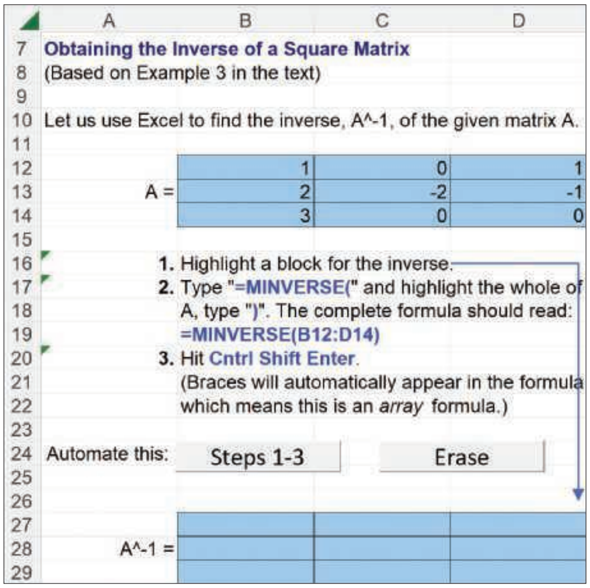
- **Interactive Tutorials** Highly interactive tutorials are included on major topics, with guided exercises that parallel the text and provide a great deal of help and feedback to assist the student.
- **Game Versions of Tutorials** These are more challenging tutorials with randomized questions that work as games (complete with "health" scores, "health vials" and an assessment of one's performance that can be sent to the instructor) and are offered alongside the traditional tutorials. These game tutorials, which mirror the traditional "more gentle" tutorials, randomize all the questions and do not give the student the answers but instead offer hints in exchange for "health points," so that just staying alive (not running out of health) can be quite challenging.
- **Adaptive Practice Sessions** The individual interactive examples in the newer game tutorials are sometimes identical to those in the WebAssign eBook, and all come equipped with adaptive practice sessions. These practice sessions may get easier if difficulties persist and present links to lists of prerequisite skills for that topic, where each in turn can offer practice topics further down in the knowledge tree. Each of these prerequisite topics includes some brief pedagogy, often with some examples.



- **Learning and Practice Modules** These are the "disembodied" interactive examples and demonstrations that populate the game tutorials and the ebook, available individually from an index page for class practice and review under "Practice and Visualize" at **AppliedMathSite.net**. Like their counterparts in the game tutorials and the ebook, these modules offer the same adaptive practice sessions.

Practice This Skill
Try Exercises 15–28.
Online: AppliedMathSite.net
Practice and visualize → Cost, revenue, and profit

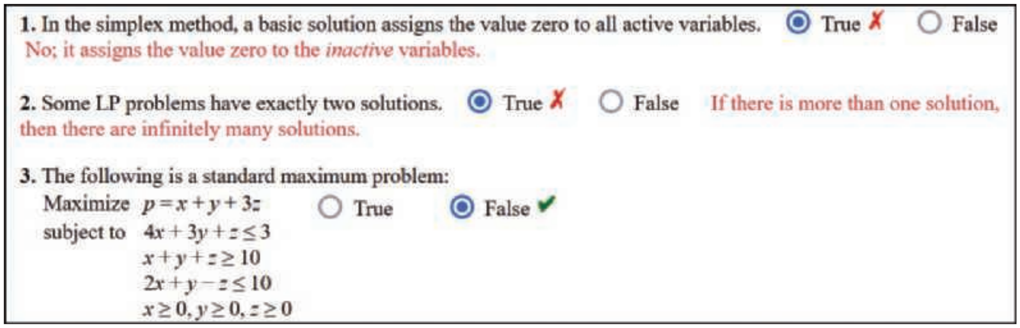
- **Videos** Section by section teaching videos that follow the online tutorials.
- **Detailed Chapter Summaries** Comprehensive summaries with randomizable interactive elements review all the basic definitions and problem-solving techniques discussed in each chapter. These are terrific pre-test study tools for students.
- **Downloadable Excel Tutorials** Detailed Excel tutorials are available for almost every section of the book. These interactive Excel sheets expand on the examples given in the text.



- **On-Line Utilities** Our collection of easy-to-use on-line utilities, referenced in the marginal notes of the textbook, allow students to solve many of the technology-based application exercises directly on the Web. The utilities available include a function grapher and evaluator that also graphs derivatives and does curve-fitting, regression tools, a time value of money calculator for annuities, a matrix algebra tool that also manipulates matrices with multinomial entries, a linear programming grapher that automatically solves two dimensional linear programming problems graphically, a powerful simplex method tool, an interactive Riemann sum grapher with a numerical integrator, and a multi-functional line entry calculator on the main page. These utilities require nothing more than a standard Web browser.

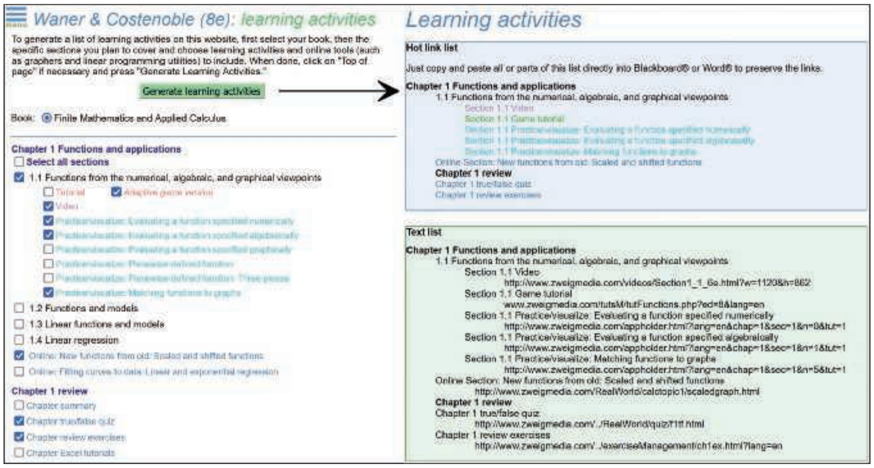
WWW.APPLIEDMATHSITE.NET
Go to the Online Utilities → Function evaluator and grapher and enter
`(x<=18)*(3.75(X-16)^2)`
`+(x>18)*(1.5(X-18)^2+15)`
for y_1. To obtain a table of values, enter the x-values 16, . . . , 21 in the Evaluator box, and press "Evaluate" at the top of the box.
Graph: Set Xmin = 15, Xmax = 22, and press "Plot Graphs".

- **Chapter True-False Quizzes** Randomized quizzes that provide feedback for many incorrect answers based on the key concepts in each chapter assist the student in further mastery of the material.



- **Supplemental Topics** We include complete interactive text and exercise sets for a selection of topics not ordinarily included in printed texts, but often requested by instructors.
- **Automatically Updating 7th Edition Homework Assignments to the 8th Edition** On the instructor's page at **AppliedMathSite.net**, the instructor can paste a comma separated list of exercises from any section of the 7th edition and have it updated to a corresponding list of (sometimes updated) exercises in the 8th edition, along with a list of exercises new to the 8th edition

- **Learning Activities Generator** Also on the instructor's page at **AppliedMathSite.net**, the instructor can automatically generate a detailed list of learning activities selected from the site (videos, tutorials, summaries, utilities, etc.) by choosing which of the features to include for one or more sections of the book and pressing "Generate Learning Activities." What results is a list of learning activities with links that can simply be pasted into the instructor's course platform; ideal for asynchronous courses.



- **Spanish** A parallel Spanish version of almost the entire web site is now deployed, allowing the user to switch languages on specific pages with a single mouse-click. In particular, all of the chapter summaries and most of the tutorials, game tutorials, and utilities are already available in Spanish.

Supplemental Resources from Cengage

Additional instructor and student resources for this product are available online. Instructor assets include a Solution and Answer Guide, Instructor's Manual, WebAssign Educator's Guide, a Transition Guide for instructors who used the previous seventh edition, PowerPoint® slides, and a test bank powered by Cognero®. Student assets include a Student Solution Manual. Sign up or sign in at **www.cengage.com** to search for and access *Finite Mathematics and Applied Calculus* and its online resources.

For Students

- **Student Solution Manual *by Waner and Costenoble*** The student solutions manual provides worked-out solutions, written by the authors, to the odd-numbered exercises in the text.

For Instructors

- **Solution and Answer Guide *by Waner and Costenoble*** The instructor's solutions manual provides worked-out solutions, written by the authors, to all exercises in the text.
- **Instructor's Manual** (new to this edition and separate from the Solution and Answer Guide) provides explanations and guidance regarding how students can benefit from using interactive media in the WebAssign eBook and directs instructors to Lecture Videos and other resources that may be useful, especially in online or hybrid courses.
- **Test Bank** Cengage's Testing Powered by Cognero is a flexible, online system that allows you to author, edit, and manage test bank content, create multiple test versions, and deliver tests from your LMS, your classroom, or wherever you choose.
- **PowerPoint® Slides** may be customized as needed or used as provided by the publisher to support lectures. **WebAssign Educator's Guide** explains the basic procedure for setting up your WebAssign course and describes the assets and resources that you can make available to students.

- **Transition Guide** contains a correlation of seventh-to-eighth edition exercise numbers to facilitate the updating of homework assignments to the new edition.
- **Guide to Teaching Online** may be helpful to instructors who are new to online teaching or who must transition from in-person teaching to an online format as may be required by local circumstances.

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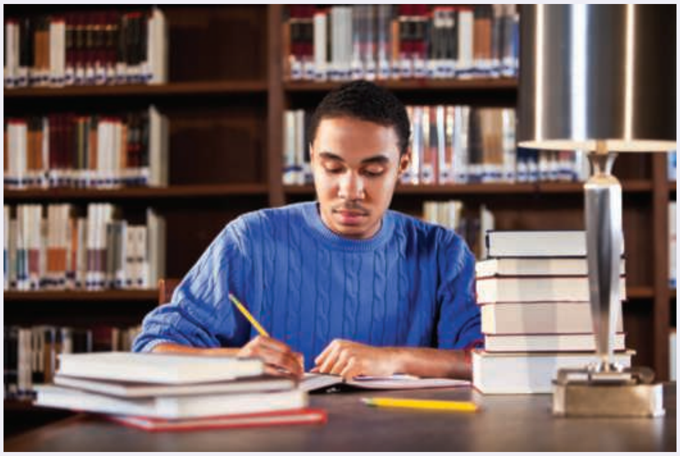
Stefan Waner
Steven R. Costenoble

0 Precalculus Review



Kali9/E+/Getty Images

www.AppliedMathSite.net
The interactive "game" tutorials at the Website give you tons of additional practice on these important topics.

Introduction

In this chapter we review some topics from algebra that you need to know to get the most out of this book. This chapter can be used either as a refresher course or as a reference.

There is one crucial fact you must always keep in mind: The letters used in algebraic expressions stand for numbers. All the rules of algebra are just facts about the arithmetic of numbers. If you are not sure whether some algebraic manipulation you are about to do is legitimate, try it first with numbers. If it doesn't work with numbers, it doesn't work.

0.1 Real Numbers

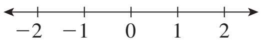
What are Real Numbers?

The **real numbers** are the numbers that can be written in decimal notation, including those that require an infinite decimal expansion. The set of real numbers includes all integers, positive, and negative; all fractions; and the irrational numbers, those with decimal expansions that never repeat. Examples of irrational numbers are

$$\sqrt{2} = 1.414213562373\ldots$$

and

$$\pi = 3.141592653589\ldots$$



The Real Line

Figure 1

It is very useful to picture the real numbers as points on a line, called the **real line**. As shown in Figure 1, larger numbers appear to the right, in the sense that if $a < b$ then the point corresponding to b is to the right of the one corresponding to a.

Q: Why are they called "real" numbers?

A: There is a larger class of numbers called *complex numbers* that include the real numbers as well as *imaginary numbers*: square roots of negative numbers. In this book we focus only on real numbers.[†] ■

[†] The names "real" and "imaginary" are unfortunate but traditional. There is nothing less real about the imaginary numbers, which have practical applications throughout the sciences and engineering.

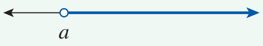
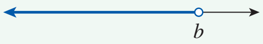
Intervals

Some subsets of the set of real numbers, called **intervals**, show up quite often and so we have a compact notation for them.

Key Concept Interval Notation

Here is a list of types of intervals along with examples.

Type	Notation	Description	Figure	Example
Closed	$[a, b]$	Set of numbers x with $a \le x \le b$	a ●——● b (includes endpoints)	$[0, 10]$
Open	(a, b)	Set of numbers x with $a < x < b$	a ○——○ b (excludes endpoints)	$(-1, 5)$
Half-Open	$(a, b]$	Set of numbers x with $a < x \le b$	a ○——● b	$(-3, 1]$
	$[a, b)$	Set of numbers x with $a \le x < b$	a ●——○ b	$[0, 5)$

Infinite	$[a, \infty)$	Set of numbers x with $a \le x$		$[10, \infty)$
	(a, ∞)	Set of numbers x with $a < x$		$(-3, \infty)$
	$(-\infty, b]$	Set of numbers x with $x \le b$		$(-\infty, -3]$
	$(-\infty, b)$	Set of numbers x with $x < b$		$(-\infty, 10)$
	$(-\infty, \infty)$	Set of all real numbers		$(-\infty, \infty)$

Operations

There are five important operations on real numbers: addition, subtraction, multiplication, division, and exponentiation. "Exponentiation" means raising a real number to a power; for instance, $3^2 = 3 \cdot 3 = 9$; $2^3 = 2 \cdot 2 \cdot 2 = 8$.

A note on technology: Most graphing calculators and spreadsheets use an asterisk * for multiplication and a caret sign ^ for exponentiation. Thus, for instance, $3 \cdot 5$ is entered as `3*5`, $3x$ as `3*x`, and 3^2 as `3^2`. On the other hand, when writing mathematics we sometimes write quantities we wish to multiply next to each other without any operation. For instance, $3x$ means $3 \cdot x$ (*multiplication by juxtaposition*) and $4(5 + x)$ means $4 \cdot (5 + x)$. This notation is called **implied multiplication**.

When we write an expression involving two or more operations, like

$$2 \cdot 3 + 4$$

or

$$\frac{2 \cdot 3^2 - 5}{4 - (-1)}$$

we need to agree on the order in which to do the operations. Does $2 \cdot 3 + 4$ mean $(2 \cdot 3) + 4 = 10$ or $2 \cdot (3 + 4) = 14$? We all agree to use the following rules for the order in which we do the operations.

Key Concept Standard Order of Operations

Parentheses and Fraction Bars

First, calculate the values of all expressions inside parentheses or brackets, working from the innermost parentheses out, before using them in other operations. In a fraction, calculate the numerator and denominator separately before doing the division.

QUICK EXAMPLES

1. $6(2 + [3 - 5] - 4) = 6(2 + (-2) - 4)$ Innermost bracket
$= 6(-4)$ Remaining expression in parentheses
$= -24.$ (Implied) multiplication

2. $\dfrac{4 - 2}{3(-2 + 1)} = \dfrac{2}{3(-1)}$ Expression in parentheses & fraction bars
$= \dfrac{2}{-3}$ Numerator and denominator separately
$= -\dfrac{2}{3}$

3. $(x + 4x)/(y + 3y) = (5x)/(4y)$‡ Expressions in parentheses

‡ We have left the $5x$ and $4y$ in parentheses as they are still expressions as opposed to single numbers (although the parentheses around the $5x$ are not needed). For instance, if $x = y = 8$ then

$$(5x)/(4y) = 40/32 = 5/4$$

whereas, as we will see below

$$5x/4y = 5 \cdot 8/4 \cdot 8 = 80.$$

Exponents

Next, perform exponentiation.

QUICK EXAMPLES

4. $2 + 4^2 = 2 + 16 = 18$ — No parentheses or fraction bars, so do exponents first.

5. $(2 + 4)^2 = 6^2$ — Expression in parentheses first
$= 36$ — Then do exponents. (Compare Quick Example 4.)

6. $2(1 + 0.1)^2 = 2(1.1)^2$ — Expression in parentheses
$= 2(1.21)$ — Exponents
$= 2.42$ — (Implied) multiplication

7. $2\left(\frac{3}{4-5}\right)^2 = 2\left(\frac{3}{-1}\right)^2$ — Expression in parentheses: fraction bars
$= 2(-3)^2$ — Finish expression in parentheses.
$= 2(9)$ — Exponents
$= 18$ — (Implied) multiplication

Multiplication and Division

Next, do all multiplications and divisions, from left to right.

QUICK EXAMPLES

8. $2(3 - 5)/4 \cdot 2 = 2(-2)/4 \cdot 2$ — Expression in parentheses
$= -4/4 \cdot 2$ — Left-most product
$= -1 \cdot 2$ — Left-most quotient
$= -2$ — Remaining product

9. $2(1 + 1/10)^2 \times 2/10 = 2(1 + 0.1)^2 \times 2/10$ — Expression in parentheses: division
$= 2(1.1)^2 \times 2/10$ — Finish expression in parentheses.
$= 2 \times 1.21 \times 2/10$ — Exponents
$= 2.42 \times 2/10$ — Left-most product
$= 4.84/10$ — Left-most product
$= 0.484$ — Remaining quotient

10. $4\,\frac{2(4-2)}{3(-2 \cdot 5)} = 4 \times \frac{2(4-2)}{3(-2 \cdot 5)}$ * — Implied multiplication
$= 4 \times \frac{2(2)}{3(-10)}$ — Expressions in parentheses: division
$= 4 \times \frac{4}{-30}$ — Fraction bars
$= \frac{16}{-30}$ — Left-most product
$= -\frac{8}{15}$ — Lowest terms

11. $-2^4 = (-1) \times 2^4$ — A negative sign before an expression means multiplication by -1.¶
$= (-1) \times 8$ — Multiplication
$= -8$

* Whereas a number written next to a fraction generally means multiplication, this is not the case with so-called **mixed fractions**, like $1\frac{3}{4}$, which means $1 + \frac{3}{4} = 1.75$. However, we will rarely use mixed fractions in this book.

¶ Spreadsheets and some programming languages would interpret -2^4 (wrongly!) as $(-2)^4 = 16$. So, when working with spreadsheets, write -2^4 as `(-1)2^4` or `(-1)*2^4` to avoid this issue.

Addition and Subtraction

Last, do all additions and subtractions, from left to right.

QUICK EXAMPLES

12. $2(3 - 5)^2 + 6 - 1 = 2(-2)^2 + 6 - 1$ — Expression in parentheses first
$= 2(4) + 6 - 1$ — Exponents
$= 8 + 6 - 1$ — Multiplication and/or division
$= 13$ — Addition and subtraction

13. $3/2 + 4 = \frac{3}{2} + 4$ — Division first

$= \frac{11}{2}$

14. $3/(2 + 4) = 3/6$ — Parentheses first.

$= \frac{1}{2}$ — Then division. Compare the preceding Quick Example.

Entering Formulas

Any good calculator or spreadsheet will respect the standard order of operations. However, we must be careful with division and exponentiation and use parentheses as necessary. The following table gives some examples of simple mathematical expressions and their equivalents in the functional format used in most graphing calculators, spreadsheets, and computer programs.

Mathematical Expression	Formula	Comments
$\frac{2}{3-x}$	`2/(3-x)`	Note the use of parentheses instead of the fraction bar. If we omit the parentheses, we get the expression shown next.
$\frac{2}{3} - x$	`2/3-x`	Without parentheses the standard order of arithmetic operations is followed.
$3xy$	`3*x*y`	Asterisks are usually used for multiplication in graphing calculators and computers.
$\frac{2}{3x}$	`2/(3*x)`	Putting the denominator in parentheses ensures that the multiplication is carried out first. Some calculators, and also some textbooks and journals, would interpret `2/3x` as $\frac{2}{3 \times x}$, giving multiplication by juxtaposition precedence over division.
$\frac{2}{3}x$	`(2/3)*x`	Putting the fraction in parentheses ensures that it is calculated first.
$\frac{2-3}{4+5}$	`(2-3)/(4+5)`	Note once again the use of parentheses in place of the fraction bar.
2^3	`2^3`	The caret ^ is commonly used to denote exponentiation.
2^{3-x}	`2^(3-x)`	Be careful to use parentheses to tell the calculator where the exponent ends. Enclose the entire exponent in parentheses.
$2^3 - x$	`2^3-x`	Without parentheses the standard order of arithmetic operations is followed.
3×2^{-4}	`3*2^(-4)`	On some calculators, the negation key is separate from the minus key.
$2^{-4\times3} \times 5$	`2^(-4*3)*5`	Note once again how parentheses enclose the entire exponent.
$100\left(1 + \frac{0.05}{12}\right)^{60}$	`100*(1+0.05/12)^60`	This is a typical calculation for compound interest.
$PV\left(1 + \frac{r}{m}\right)^{mt}$	`PV*(1+r/m)^(m*t)`	This is the compound interest formula. *PV* is understood to be a single number (present value) and not the product of *P* and *V* (or else we would have used `P*V`).
$\frac{2^{3-2} \times 5}{y - x}$	`2^(3-2)*5/(y-x)` or `(2^(3-2)*5)/(y-x)`	Notice again the use of parentheses to hold the denominator together. We could also have enclosed the numerator in parentheses, although this is optional. (Why?)
$\frac{2^y + 1}{2 - 4^{3x}}$	`(2^y+1)/(2-4^(3*x))`	Here, it is necessary to enclose both the numerator and the denominator in parentheses.
$2^y + \frac{1}{2} - 4^{3x}$	`2^y+1/2-4^(3*x)`	This is the effect of leaving out the parentheses around the numerator and denominator in the previous expression.

Accuracy and Rounding

When we use a calculator or computer, the results of our calculations are often given to far more decimal places than are useful. For example, suppose we are told that a square has an area of 2.0 square feet and we are asked how long its sides are. Each side is the square root of the area, which the calculator tells us is

$$\sqrt{2} \approx 1.414213562\ldots$$

However, the measurement of 2.0 square feet is probably accurate to only two digits, so our estimate of the lengths of the sides can be no more accurate than that. Therefore, we round the answer to two digits:

$$\text{Length of one side} \approx 1.4 \text{ feet.}$$

The digits that follow 1.4 are meaningless. The following guide makes these ideas more precise.

Key Concept Significant Digits, Decimal Places, and Rounding

The number of **significant digits** in a decimal representation of a number is the number of digits that are not leading zeros after the decimal point (as in .0005) or trailing zeros before the decimal point (as in 5,400,000). We say that a value is **accurate to *n* significant digits** if only the first n significant digits are meaningful.

When to Round

After doing a computation in which all the quantities are accurate to no more than n significant digits, round the final result to n significant digits.

QUICK EXAMPLES

15. 0.00067 has two significant digits. — The 000 before 67 are leading zeros.

16. 0.000670 has three significant digits. — The 0 after 67 is significant.

17. 5,400,000 has two or more significant digits. — We can't say how many of the zeros are trailing.*

18. 5,400,001 has 7 significant digits. — The string of zeros is not trailing.

19. Rounding 63,918 to three significant digits gives 63,900.

20. Rounding 63,958 to three significant digits gives 64,000.

21. $\pi = 3.141592653\ldots,\ \frac{22}{7} = 3.142857142\ldots.$
Therefore, $\frac{22}{7}$ is an approximation of π that is accurate to only three significant digits (3.14).

22. $4.02(1 + 0.02)1.4 \approx 4.13$ — We rounded to three significant digits.

* If we obtained 5,400,000 by rounding 5,401,011, then it has three significant digits because the zero after the 4 is significant. On the other hand, it we obtained it by rounding 5,411,234, then it has only two significant digits. The use of scientific notation avoids this ambiguity: 5.40×10^6 (or `5.40E6` on a calculator or computer) is accurate to three digits and 5.4×10^6 is accurate to two.

One more point, though: If, in a long calculation, you round the intermediate results, your final answer may be even less accurate than you think. As a general rule,

When calculating, don't round intermediate results. Rather, use the most accurate results obtainable or have your calculator or computer store them for you.

When you are done with the calculation, then round your answer to the appropriate number of digits of accuracy.

0.1 Exercises

Calculate each expression in Exercises 1–24, giving the answer as a whole number or a fraction in lowest terms.

1. $2(4+(-1))(2\cdot -4)$

2. $3+([4-2]\cdot 9)$

3. `20/(3*)-1`

4. `2-(3*4)/10`

5. $\dfrac{3+([3+(-5)])}{3-2\times 2}$

6. $\dfrac{12-(1-4)}{2(5-1)\cdot 2-1}$

7. `(2-5*(-1))/1-2*(-1)`

8. `2-5*(-1)/(1-2*(-1))`

9. $2\cdot(-1)^2/2$

10. $2+4\cdot 3^2$

11. $2\cdot 4^2+1$

12. $1-3\cdot(-2)^2$

13. `3^2+2^2+1`

14. `32^(2^2-2)`

15. $\dfrac{3-2(-3)^2}{-6(4-1)^2}$

16. $\dfrac{1-2(1-4)^2}{2(5-1)^2\cdot 2}$

17. `10*(1+1/10)^3`

18. `121/(1+1/10)^2`

19. $3\left[\dfrac{-2\cdot 3^2}{-(4-1)^2}\right]$

20. $-\left[\dfrac{8(1-4)^2}{-9(5-1)^2}\right]$

21. $3\left[1-\left(-\dfrac{1}{2}\right)^2\right]^2+1$

22. $3\left[\dfrac{1}{9}-\left(\dfrac{2}{3}\right)^2\right]^2+1$

23. `(1/2)^2-1/2^2`

24. `2/(1^2)-(2/1)^2`

Convert each expression in Exercises 25–50 into its technology formula equivalent as in the table in the text.

25. $3\times(2-5)$

26. $4+\dfrac{5}{9}$

27. $\dfrac{3}{2-5}$

28. $\dfrac{4-1}{3}$

29. $\dfrac{3-1}{8+6}$

30. $3+\dfrac{3}{2-9}$

31. $3-\dfrac{4+7}{8}$

32. $\dfrac{4\times 2}{\left(\frac{2}{3}\right)}$

33. $\dfrac{2}{3+x}-xy^2$

34. $3+\dfrac{3+x}{xy}$

35. $3.1x^3-4x^{-2}-\dfrac{60}{x^2-1}$

36. $2.1x^{-3}-x^{-1}+\dfrac{x^2-3}{2}$

37. $\dfrac{\left(\frac{2}{3}\right)}{5}$

38. $\dfrac{2}{\left(\frac{3}{5}\right)}$

39. $3^{4-5}\times 6$

40. $\dfrac{2}{3+5^{7-9}}$

41. $3\left(1+\dfrac{4}{100}\right)^{-3}$

42. $3\left(\dfrac{1+4}{100}\right)^{-3}$

43. $3^{2x-1}+4^x-1$

44. $2^{x^2}-(2^{2x})^2$

45. 2^{2x^2-x+1}

46. $2^{2x^2-x}+1$

47. $\dfrac{4e^{-2x}}{2-3e^{-2x}}$

48. $\dfrac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$

49. $3\left(1-\left(-\dfrac{1}{2}\right)^2\right)^2+1$

50. $3\left(\dfrac{1}{9}-\left(\dfrac{2}{3}\right)^2\right)^2+1$

0.2 Exponents and Radicals

Integer Exponents

In Section 0.1 we mentioned exponentiation, or "raising to a power"; for example, $2^3=2\cdot 2\cdot 2$. In this section we discuss the algebra of exponentials more fully. First, we look at *integer* exponents: cases in which the powers are positive or negative whole numbers.

Key Concept Positive Integer Exponents

If a is any real number and n is any positive integer, then by a^n we mean the quantity $a\cdot a\cdot\cdots\cdot a$ (n times); thus, $a^1=a$, $a^2=a\cdot a$, $a^5=a\cdot a\cdot a\cdot a\cdot a$. In the expression a^n the number n is called the **exponent**, and the number a is called the **base**.

QUICK EXAMPLES

1. $3^2 = 9$

2. $2^3 = 8$

3. $0^{24} = 0$

4. $(-1)^5 = -1$

5. $10^3 = 1{,}000$

6. $10^5 = 100{,}000$

Key Concept Negative Integer Exponents

If a is any real number other than zero and n is any positive integer, then we define

$$a^{-n} = \frac{1}{a^n} = \frac{1}{a \cdot a \cdot \cdots \cdot a} \quad (n \text{ times})$$

QUICK EXAMPLES

7. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

8. $1^{-27} = \frac{1}{1^{27}} = 1$

9. $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$

10. $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{9}$

11. $y^7 y^{-2} = y^7 \frac{1}{y^2}$

12. 0^{-2} is not defined.

13. $\frac{1}{3^{-2}} = \frac{1}{1/9} = 9$

14. $\frac{1}{a^{-n}} = \frac{1}{1/a^n} = a^n$ for positive integers n†

† Thus, the rule

$$a^{-n} = \frac{1}{a^n}$$

works for both positive and negative n: Moving an expression of the form a^n from the numerator to the denominator (or vice-versa) changes the sign of the exponent.

Key Concept Zero Exponent

If a is any real number other than zero, then we define

$$a^0 = 1$$

QUICK EXAMPLES

15. $3^0 = 1$

16. $1{,}000{,}000^0 = 1$

17. 0^0 is not defined.

When combining exponential expressions, we use the following identities.

Key Concept Exponent Identities

The following identities hold for all integer exponents and nonzero bases.

Identity	QUICK EXAMPLES
1. $a^m a^n = a^{m+n}$	$2^3 2^2 = 2^{3+2} = 2^5 = 32$
	$x^3 x^{-4} = x^{3-4} = x^{-1} = \dfrac{1}{x}$
	$\dfrac{x^3}{x^{-2}} = x^3 \dfrac{1}{x^{-2}} = x^3 x^2 = x^5$
2. $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{4^3}{4^2} = 4^{3-2} = 4^1 = 4$
	$\dfrac{x^3}{x^{-2}} = x^{3-(-2)} = x^5$
	$\dfrac{3^2}{3^4} = 3^{2-4} = 3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$
3. $(a^n)^m = a^{mn}$	$(2^2)^3 = 2^6 = 64$
	$(3^x)^2 = 3^{2x}$
4. $(ab)^n = a^n b^n$	$(4 \cdot 2)^2 = 4^2 2^2 = 64$
	$(-2y)^4 = (-2)^4 y^4 = 16y^4$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{4}{3}\right)^2 = \dfrac{4^2}{3^2} = \dfrac{16}{9}$
	$\left(\dfrac{x}{-y}\right)^3 = \dfrac{x^3}{(-y)^3} = -\dfrac{x^3}{y^3}$

Caution

- In the first two identities, the bases of the expressions must be the same. For example, the first gives $3^2 3^4 = 3^6$, but does not apply to $3^2 4^2$.
- People sometimes invent their own identities, such as $a^m + a^n = a^{m+n}$, which is wrong! If you wind up with something like $2^3 + 2^4$, you are stuck with it; there are no identities around to simplify it further.‡

‡ You might factor out 2^3, but whether that would be a simplification depends on what you were going to do with the expression next. Try it with $a = m = n = 1$.

EXAMPLE 1 Combining the Identities

$$\frac{(x^2)^3}{x^3} = \frac{x^6}{x^3} \quad \text{By (3)}$$
$$= x^{6-3} \quad \text{By (2)}$$
$$= x^3.$$

$$\frac{(x^4 y)^3}{y} = \frac{(x^4)^3 y^3}{y} \quad \text{By (4)}$$
$$= \frac{x^{12} y^3}{y} \quad \text{By (3)}$$
$$= x^{12} y^{3-1} \quad \text{By (2)}$$
$$= x^{12} y^2.$$