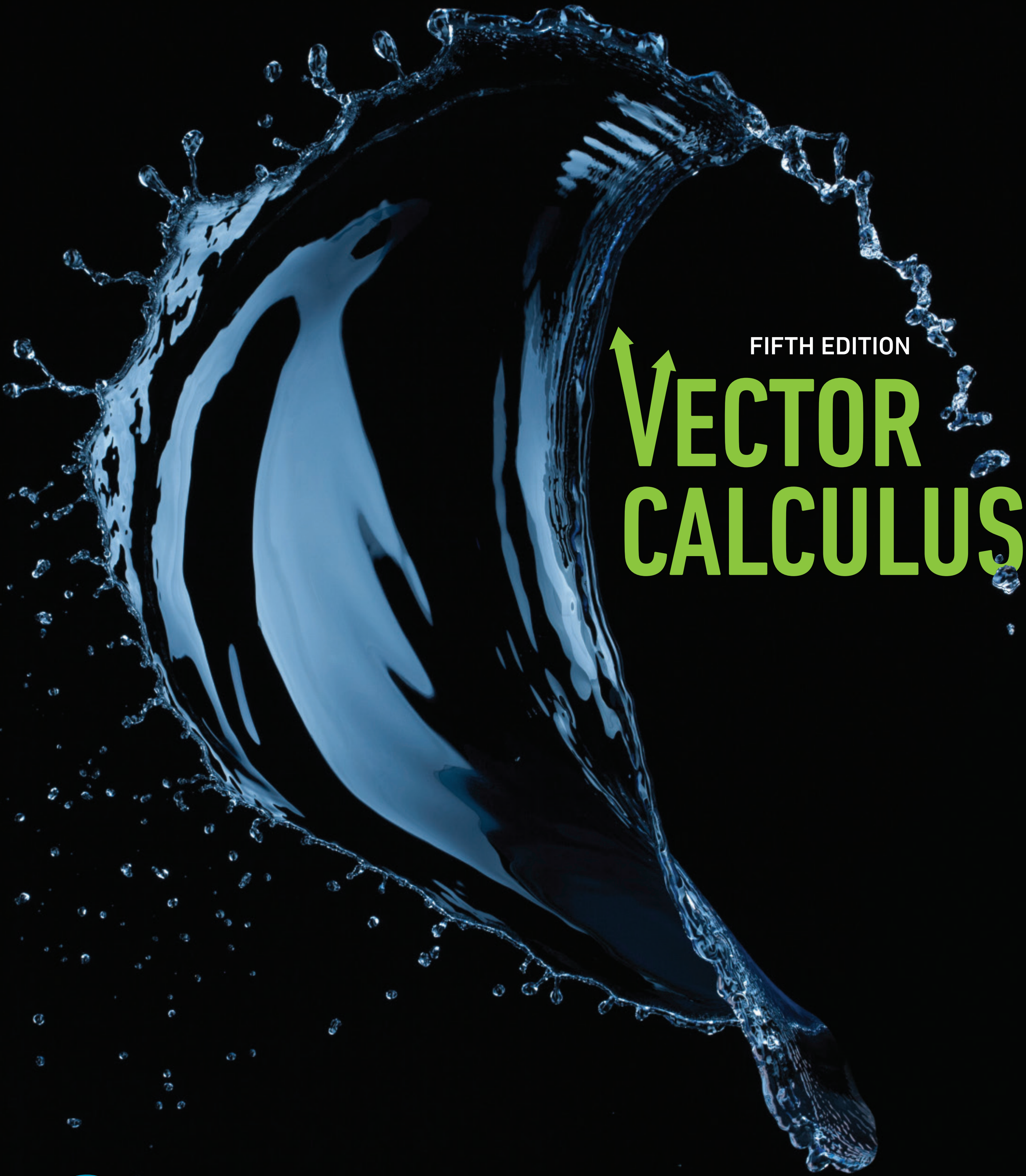

FIFTH EDITION

# VECTOR CALCULUS

SUSAN JANE COLLEY • SANTIAGO CAÑEZ

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FIFTH EDITION

# Vector Calculus

Susan Jane Colley

*Oberlin College*

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Pearson

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To Will and Diane, with love
—SJC

To Adelina, Addy, Olivia, and Cecily
—SC

# About the Authors

## Susan Jane Colley


*William Colley*

Susan Colley is the Andrew and Pauline Delaney Professor of Mathematics at Oberlin College. She has served as chair of the department as well as editor of *The American Mathematical Monthly*.

She received S.B. and Ph.D. degrees in mathematics from the Massachusetts Institute of Technology prior to joining the faculty at Oberlin in 1983.

Her research has primarily focused on algebraic geometry, particularly enumerative problems, multiple-point singularities, and higher-order data and contact of plane curves.

Professor Colley has published papers on algebraic geometry and commutative algebra as well as articles on other mathematical subjects. She has lectured internationally on her research and has taught a wide range of subjects in undergraduate mathematics.

Professor Colley is a member of several professional and honorary societies, including the American Mathematical Society, the Mathematical Association of America, Phi Beta Kappa, and Sigma Xi.

## Santiago Cañez

Santiago Cañez is currently a Charles Deering McCormick Distinguished Professor of Instruction in the Department of Mathematics at Northwestern University and serves as its director of undergraduate studies. He received a B.S. degree in mathematics from the University of Arizona and a Ph.D. degree in mathematics from the University of California at Berkeley prior to joining the faculty at Northwestern in 2012.

Professor Cañez works in symplectic geometry and mathematical physics and is in particular interested in groupoids and higher-order structures. He has taught a wide range of courses and has supervised numerous student theses and research projects. He is a member of the American Mathematical Society and the Mathematical Association of America.

# Contents

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# Preface

Physical and natural phenomena depend on a complex array of factors. The sociologist or psychologist who studies group behavior, the economist who endeavors to understand the vagaries of a nation's employment cycles, the physicist who observes the trajectory of a particle or planet, or indeed anyone who seeks to understand geometry in two, three, or more dimensions recognizes the need to analyze changing quantities that depend on more than a single variable. Vector calculus is the essential mathematical tool for such analysis. Moreover, it is an exciting and beautiful subject in its own right, a true adventure in many dimensions.

The only technical prerequisite for this text, which is intended for a sophomore-level course in multivariable calculus, is a standard course in the calculus of functions of one variable. In particular, the necessary matrix arithmetic and algebra (not linear algebra) are developed as needed. Although the mathematical background assumed is not exceptional, the reader will still be challenged in places.

Our objectives in writing the book are simple ones: to develop in students a sound conceptual grasp of vector calculus and to help them begin the transition from first-year calculus to more advanced technical mathematics. We believe that the first goal can be met, at least in part, through the use of vector and matrix notation, so that many results, especially those of differential calculus, can be stated with reasonable levels of clarity and generality. Properly described, results in the calculus of several variables can look quite similar to those of the calculus of one variable. Reasoning by analogy will thus be an important pedagogical tool. We also believe that a conceptual understanding of mathematics can be obtained through the development of a good geometric intuition. Although many results are stated in the case of $n$ variables (where $n$ is arbitrary), we recognize that the most important and motivational examples usually arise for functions of two and three variables, so these concrete and visual situations are emphasized to explicate the general theory. Vector calculus is in many ways an ideal subject for students to begin exploration of the interrelations among analysis, geometry, and matrix algebra.

Multivariable calculus, for many students, represents the beginning of significant mathematical maturation. Consequently, we have written a rather expansive text so that they can see that there is a story behind the results, techniques, and examples—that the subject coheres and that this coherence is important for problem solving. To indicate some of the power of the methods introduced, a number of topics, not always discussed very fully in a first multivariable calculus course, are treated here in some detail:

- an early introduction of cylindrical and spherical coordinates (§1.7);
- the use of vector techniques to derive Kepler's laws of planetary motion (§3.1);
- the elementary differential geometry of curves in $\mathbf{R}^3$, including discussion of curvature, torsion, and the Frenet–Serret formulas for the moving frame (§3.2);
- Taylor's formula for functions of several variables (§4.1);
- the use of the Hessian matrix to determine the nature (as local extrema) of critical points of functions of $n$ variables (§4.2 and §4.3);
- an extended discussion of the change of variables formula in double and triple integrals (§5.5);
- applications of vector analysis to physics (§7.4);
- an introduction to differential forms and the generalized Stokes's theorem (Chapter 8).

Included are a number of proofs of important results. The more technical proofs are collected as addenda at the ends of the appropriate sections so as not to disrupt the main conceptual flow and to allow for greater flexibility of use by the instructor and student. Nonetheless, some proofs (or sketches of proofs) embody such central ideas that they are included in the main body of the text.

## New in the Fifth Edition

We have retained the overall structure and tone of prior editions. New features in this edition include the following:

- **NEW:** For the first time, this text is available as a Pearson eText, featuring a number of interactive GeoGebra applets.
- clarifications, new examples, and new exercises throughout the text;
- new derivations of the orthogonal projection formula (§1.3) and the Cauchy–Schwarz inequality (§1.6);
- a description of the geometric interpretation of second-order partial derivatives (§2.4);
- a description of the interpretation of the Lagrange multiplier (§4.3);
- new terminology in Chapter 5 to describe elementary regions of integration, and more examples of setting up double and triple integrals;
- a new subsection in §5.6 on probability as an application of multiple integrals, and new miscellaneous exercises in Chapter 5 on expected value;
- new examples illustrating interesting uses of Green's theorem (§6.2);
- new miscellaneous exercises in Chapters 1 and 4 for readers more familiar with linear algebra.
- **Authors' DEI statement:** We conducted an external review of the text's content to determine how it could be improved to address issues related to diversity, equity, and inclusion. The results of that review informed the revision.

## How to Use This Book

There is more material in this book than can be covered comfortably during a single semester. Hence, the instructor will wish to eliminate some topics or subtopics—or to abbreviate the rather leisurely presentations of limits and differentiability. Since some instructors may find themselves without the time to treat surface integrals in detail, we have separated all material concerning parametrized surfaces, surface integrals, and Stokes's and Gauss's theorems (Chapter 7) from that concerning line integrals and Green's theorem (Chapter 6). In particular, in a one-semester course for students having little or no experience with vectors or matrices, instructors can probably expect to cover most of the material in Chapters 1–6, although no doubt it will be necessary to omit some of the optional subsections and to downplay many of the proofs of results. A rough outline for such a course, allowing for some instructor discretion, could be the following:

| | |
|---|---|
| Chapter 1 | 8–9 lectures |
| Chapter 2 | 9 lectures |
| Chapter 3 | 4–5 lectures |
| Chapter 4 | 5–6 lectures |
| Chapter 5 | 8 lectures |
| Chapter 6 | 4 lectures |
| | 38–41 lectures |

If students have a richer background (so that much of the material in Chapter 1 can be left largely to them to read on their own), then it should be possible to treat a good portion of Chapter 7 as well. For a two-quarter or two-semester course, it should be possible to work through the entire book with reasonable care and rigor, although coverage of Chapter 8 should depend on students' exposure to introductory linear algebra, as somewhat more sophistication is assumed there.

The exercises vary from relatively routine computations to more challenging and provocative problems, generally (but not invariably) increasing in difficulty within each section. In a number of instances, groups of problems serve to introduce supplementary topics or new applications. Each chapter concludes with a set of miscellaneous exercises that both review and extend the ideas introduced in the chapter.

A word about the use of technology. The text was written without reference to any particular computer software or graphing calculator. Most of the exercises can be solved by hand, although there is no reason not to turn over some of the more tedious calculations to a computer. Those exercises that *require* a computer for computational or graphical purposes are marked with the symbol T and should be amenable to software such as *Mathematica*®, Maple®, or MATLAB.

## Ancillary Materials

An **Instructor's Solutions Manual**, containing complete solutions to all of the exercises, is available to course instructors from the Pearson Instructor Resource Center (www.pearsonhighered.com/irc), as are many Microsoft® PowerPoint® files and Wolfram *Mathematica*® notebooks that can be adapted for classroom use. The reader can find errata for the text and accompanying solutions manuals at the following address: www.oberlin.edu/math/faculty/colley/VCErrata.html

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We are very grateful to many individuals for sharing their thoughts and ideas about multivariable calculus. In particular, Susan Colley expresses particular appreciation to her Oberlin colleagues (past and present) Bob Geitz, Kevin Hartshorn, Michael Henle (who, among other things, carefully read the draft of Chapter 8), Gary Kennedy, Dan King, Greg Quenell, Michael Raney, Daniel Steinberg, Daniel Styer, Richard Vale, Jim Walsh, and Elizabeth Wilmer for their conversations with her. She is also grateful to John Alongi, Northwestern University; Matthew Conner, University of California, Davis; Pisheng Ding, Illinois State University; Henry C. King, University of Maryland; Stephen B. Maurer, Swarthmore College; Karen Saxe, Macalester College; David Singer, Case Western Reserve University; and Mark R. Treuden, University of Wisconsin at Stevens Point, for their helpful comments. Santiago Cañez would in particular like to thank his Northwestern colleagues Aaron Peterson and Daniel Cuzzocreo for helpful suggestions. Several colleagues reviewed various versions of the manuscript, and we are happy to acknowledge their efforts and many fine suggestions. In particular, for the first four editions, Susan Colley thanks the following reviewers:

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# To the Student: Some Preliminary Notation

Here are the ideas that you need to keep in mind as you read this book and learn vector calculus.

Given two sets $A$ and $B$, we assume that you are familiar with the notation $A \bigcup B$ for the **union** of $A$ and $B$—those elements that are in either $A$ or $B$ (or both):

$$A \bigcup B = \{x | x \in A \text{ or } x \in B\}.$$

Similarly, $A \bigcap B$ is used to denote the **intersection** of $A$ and $B$—those elements that are in both $A$ and $B$:

$$A \bigcap B = \{x | x \in A \text{ and } x \in B\}.$$

The notation $A \subseteq B$, or $A \subset B$, indicates that $A$ is a **subset** of $B$ (possibly empty or equal to $B$).

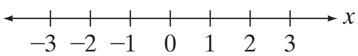



**FIGURE 1** The coordinate line **R**.

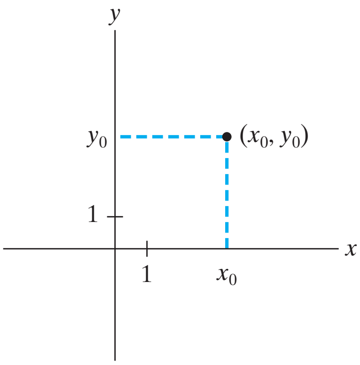



**FIGURE 2** The coordinate plane $\mathbf{R}^2$.

One-dimensional space (also called the **real line** or **R**) is just a straight line. We put real number coordinates on this line by placing negative numbers on the left and positive numbers on the right. (See Figure 1.)

Two-dimensional space, denoted $\mathbf{R}^2$, is the familiar Cartesian plane. If we construct two perpendicular lines (the $x$- and $y$-**coordinate axes**), set the **origin** as the point of intersection of the axes, and establish numerical scales on these lines, then we may locate a point in $\mathbf{R}^2$ by giving an ordered pair of numbers $(x, y)$, the **coordinates** of the point. Note that the coordinate axes divide the plane into four **quadrants**. (See Figure 2.)

Three-dimensional space, denoted $\mathbf{R}^3$, requires three mutually perpendicular coordinate axes (called the $x$-, $y$- and $z$-**axes**) that meet in a single point (called the **origin**) in order to locate an arbitrary point. Analogous to the case of $\mathbf{R}^2$, if we establish scales on the axes, then we can locate a point in $\mathbf{R}^3$ by giving an ordered triple of numbers $(x, y, z)$. The coordinate axes divide three-dimensional space into eight **octants**. It takes some practice to get your sense of perspective correct when sketching points in $\mathbf{R}^3$. (See Figure 3.) Sometimes we draw the coordinate axes in $\mathbf{R}^3$ in different orientations in order to get a better view of things. However, we always maintain the axes in a **right-handed configuration**. This means that if you curl the fingers of your right hand from the positive $x$-axis to the positive $y$-axis, then your thumb will point along the positive $z$-axis. (See Figure 4.)

Although you need to recall particular techniques and methods from the calculus you have already learned, here are some of the more important concepts to keep in mind: Given a function $f(x)$, the **derivative** $f'(x)$ is the limit (if it exists) of the difference quotient of the function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The significance of the derivative $f'(x_0)$ is that it measures the slope of the line tangent to the graph of $f$ at the point $(x_0, f(x_0))$. (See Figure 5.) The derivative may also be considered to give the instantaneous rate of change of $f$ at $x = x_0$. We also denote the derivative $f'(x)$ by $df/dx$.

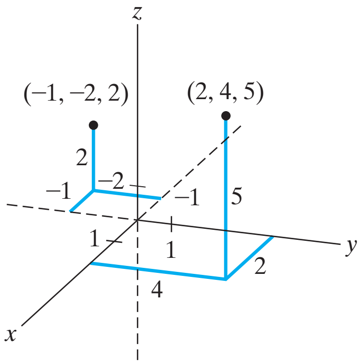



**FIGURE 3** Three-dimensional space $\mathbf{R}^3$. Selected points are graphed.

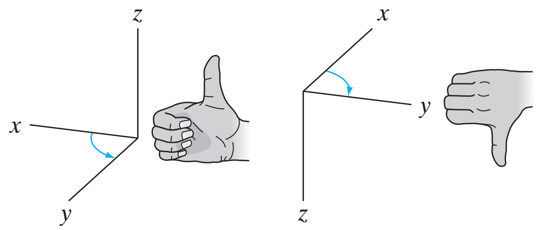



**FIGURE 4** The $x$-, $y$-, and $z$-axes in $\mathbf{R}^3$ are always drawn in a right-handed configuration.

The **definite integral** $\int_a^b f(x)\,dx$ of $f$ on the closed interval $[a, b]$ is the limit (provided it exists) of the so-called **Riemann sums** of $f$:

$$\int_a^b f(x)\,dx = \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

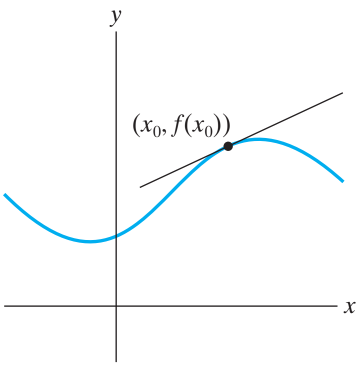



**FIGURE 5** The derivative $f'(x_0)$ is the slope of the tangent line to $y = f(x)$ at $(x_0, f(x_0))$.

Here $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ denotes a **partition** of $[a, b]$ into subintervals $[x_{i-1}, x_i]$, the symbol $\Delta x_i = x_i - x_{i-1}$ (the length of the subinterval), and $x_i^*$ denotes any point in $[x_{i-1}, x_i]$. If $f(x) \geq 0$ on $[a, b]$, then each term $f(x_i^*)\Delta x_i$ in the Riemann sum is the area of a rectangle related to the graph of $f$. The Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ thus approximates the total area under the graph of $f$ between $x = a$ and $x = b$. (See Figure 6.)

The definite integral $\int_a^b f(x)\,dx$, if it exists, is taken to represent the area under $y = f(x)$ between $x = a$ and $x = b$. (See Figure 7.)

The derivative and the definite integral are connected by an elegant result known as **the fundamental theorem of calculus**. Let $f(x)$ be a continuous

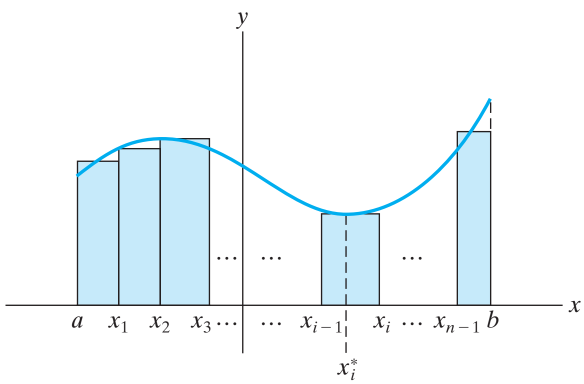



**FIGURE 6** If $f(x) \geq 0$ on $[a, b]$, then the Riemann sum approximates the area under $y = f(x)$ by giving the sum of areas of rectangles.

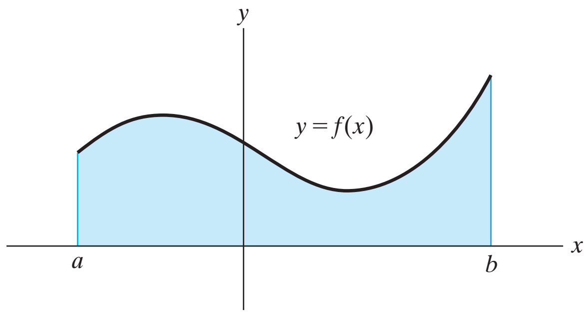



**FIGURE 7** The area under the graph of $y = f(x)$ is $\int_a^b f(x)\,dx$.

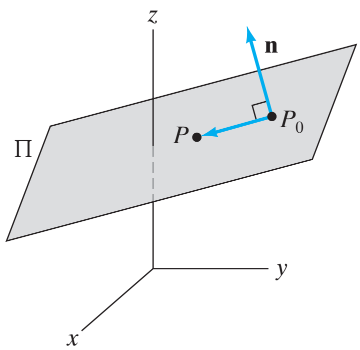
function of one variable, and let $F(x)$ be such that $F'(x) = f(x)$. (The function $F$ is called an **antiderivative** of $f$.) Then

**1.** $\displaystyle\int_a^b f(x)dx = F(b) - F(a);$

**2.** $\displaystyle\frac{d}{dx}\int_a^x f(t)dt = f(x).$

Finally, the end of an example is denoted by the symbol ▭ and the beginning and end of a proof by the symbol ■.

# 1 Vectors




The idea of describing space in terms of coordinates played a major role in the development of mathematics and led to the ability to describe planes, spheres, and other geometric objects in terms of equations. In this chapter, we develop the tools necessary to formulate such equations, with the concept of a vector playing a key role. In the accompanying figure for example, we see that formulating an equation that characterizes points on a plane in space (such as a plane tangent to a sphere) requires only knowledge of a given point on the plane and a vector perpendicular to the plane.



## 1.1 Vectors in Two and Three Dimensions

For your study of the calculus of several variables, the notion of a vector is fundamental. As is the case for many of the concepts we shall explore, there are both *algebraic* and *geometric* points of view. You should become comfortable with both perspectives in order to solve problems effectively and to build on your basic understanding of the subject.

### Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: The Algebraic Notion

**DEFINITION 1.1** A **vector** in $\mathbf{R}^2$ is simply an ordered pair of real numbers. That is, a vector in $\mathbf{R}^2$ may be written as

$$(a_1, a_2) \quad (\text{e.g., } (1, 2) \text{ or } (\pi, 17)).$$

Similarly, a **vector** in $\mathbf{R}^3$ is simply an ordered triple of real numbers. That is, a vector in $\mathbf{R}^3$ may be written as

$$(a_1, a_2, a_3) \quad (\text{e.g., } (\pi, e, \sqrt{2})).$$

To emphasize that we want to consider the pair or triple of numbers as a single unit, we will use **boldface** letters; hence $\mathbf{a} = (a_1, a_2)$ or $\mathbf{a} = (a_1, a_2, a_3)$ will be our standard notation for vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$. Whether we mean that $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or in $\mathbf{R}^3$ will be clear from context (or else won't be important to the discussion). When doing handwritten work, it is difficult to "boldface" anything, so you'll want to put an arrow over the letter. Thus, $\vec{a}$ will mean the same thing as $\mathbf{a}$. Whatever notation you decide to use, it's important that you distinguish the *vector* $\mathbf{a}$ (or $\vec{a}$) from the *single real number* $a$. To contrast them with vectors, we will also refer to single real numbers as **scalars**.

In order to do anything interesting with vectors, it's necessary to develop some arithmetic operations for working with them. Before doing this, however, we need to know when two vectors are equal.

**DEFINITION 1.2** Two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ in $\mathbf{R}^2$ are **equal** if their corresponding components are equal, that is, if $a_1 = b_1$ and $a_2 = b_2$. The same definition holds for vectors in $\mathbf{R}^3$: $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are **equal** if their corresponding components are equal, that is, if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

**EXAMPLE 1** The vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (\frac{3}{3}, \frac{6}{3})$ are equal in $\mathbf{R}^2$, but $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, 3, 1)$ are *not* equal in $\mathbf{R}^3$.

Next, we discuss the operations of vector addition and scalar multiplication. We'll do this by considering vectors in $\mathbf{R}^3$ only; exactly the same remarks will hold for vectors in $\mathbf{R}^2$ if we simply ignore the last component.

**DEFINITION 1.3 (Vector addition)** Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in $\mathbf{R}^3$. Then the **vector sum** $\mathbf{a} + \mathbf{b}$ is the vector in $\mathbf{R}^3$ obtained via componentwise addition: $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

**EXAMPLE 2** We have $(0, 1, 3) + (7, -2, 10) = (7, -1, 13)$ and (in $\mathbf{R}^2$):

$$(1, 1) + (\pi, \sqrt{2}) = (1 + \pi, 1 + \sqrt{2}).$$

**Properties of vector addition.** We have

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ for all $\mathbf{a}$, $\mathbf{b}$ in $\mathbf{R}^3$ (commutativity);
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in $\mathbf{R}^3$ (associativity);
3. a special vector, denoted $\mathbf{0}$ (and called the **zero vector**), with the property that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a}$ in $\mathbf{R}^3$.

These three properties require proofs, which, like most facts involving the algebra of vectors, can be obtained by explicitly writing out the vector components. For example, for property 1, we have that if

$$\mathbf{a} = (a_1, a_2, a_3) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, b_3),$$

then

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \\ &= (b_1 + a_1, b_2 + a_2, b_3 + a_3) \\ &= \mathbf{b} + \mathbf{a},\end{aligned}$$

since real number addition is commutative. For property 3, the "special vector" is just the vector whose components are all zero: $\mathbf{0} = (0, 0, 0)$. It's then easy to check that property 3 holds by writing out components. Similarly for property 2, so we leave the details as exercises.

**DEFINITION 1.4 (Scalar multiplication)** Let $\mathbf{a} = (a_1, a_2, a_3)$ be a vector in $\mathbf{R}^3$ and let $k \in \mathbf{R}$ be a scalar (real number). Then the **scalar product** $k\mathbf{a}$ is the vector in $\mathbf{R}^3$ given by multiplying each component of $\mathbf{a}$ by $k$: $k\mathbf{a} = (ka_1, ka_2, ka_3)$.

**EXAMPLE 3** If $\mathbf{a} = (2, 0, \sqrt{2})$ and $k = 7$, then $k\mathbf{a} = (14, 0, 7\sqrt{2})$.

The results that follow are not difficult to check—just write out the vector components.

**Properties of scalar multiplication.** For all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^3$ (or $\mathbf{R}^2$) and scalars $k$ and $l$ in $\mathbf{R}$, we have

1. $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$ (distributivity);
2. $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ (distributivity);
3. $k(l\mathbf{a}) = (kl)\mathbf{a} = l(k\mathbf{a})$.

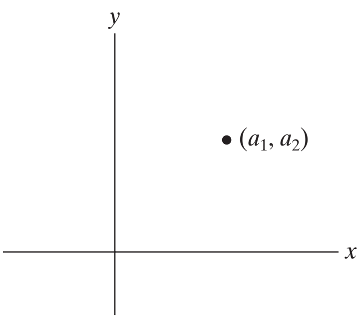



**FIGURE 1.1** A vector $\mathbf{a} \in \mathbf{R}^2$ corresponds to a point in $\mathbf{R}^2$.

It is worth remarking that none of these definitions or properties really depends on dimension, that is, on the number of components. Therefore we could have introduced the algebraic concept of a vector in $\mathbf{R}^n$ as an **ordered** $\boldsymbol{n}$**-tuple** $(a_1, a_2, \ldots, a_n)$ of real numbers and defined addition and scalar multiplication in a way analogous to what we did for $\mathbf{R}^2$ and $\mathbf{R}^3$. Think about what such a generalization means. We will discuss some of the technicalities involved in §1.6.

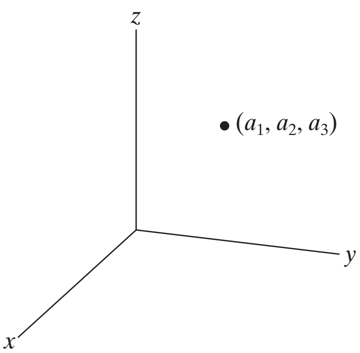



**FIGURE 1.2** A vector $\mathbf{a} \in \mathbf{R}^3$ corresponds to a point in $\mathbf{R}^3$.

## Vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$: The Geometric Notion

Although the algebra of vectors is certainly important and you should become adept at working algebraically, the formal definitions and properties tend to present a rather sterile picture of vectors. A better motivation for the definitions just given comes from geometry. We explore this geometry now. First of all, the fact that a vector $\mathbf{a}$ in $\mathbf{R}^2$ is a pair of real numbers $(a_1, a_2)$ should make you think of the coordinates of a point in $\mathbf{R}^2$. (See Figure 1.1.) Similarly, if $\mathbf{a} \in \mathbf{R}^3$, then $\mathbf{a}$ may be written as $(a_1, a_2, a_3)$, and this triple of numbers may be thought of as the coordinates of a point in $\mathbf{R}^3$. (See Figure 1.2.)

All of this is fine, but the results of performing vector addition or scalar multiplication don't have very interesting or meaningful geometric interpretations in terms of points. As we shall see, it is better to visualize a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$ as an arrow that begins at the origin and ends at the point. (See Figure 1.3.)

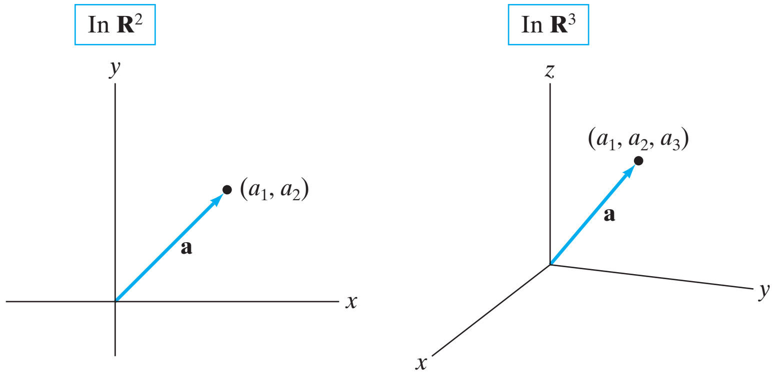



**FIGURE 1.3** A vector $\mathbf{a}$ in $\mathbf{R}^2$ or $\mathbf{R}^3$ is represented by an arrow from the origin to $\mathbf{a}$.

Such a depiction is often referred to as the **position vector** of the point $(a_1, a_2)$ or $(a_1, a_2, a_3)$.

If you've studied vectors in physics, you have heard them described as objects having "magnitude and direction." Figure 1.3 demonstrates this concept, provided that we take "magnitude" to mean "length of the arrow" and "direction" to be the orientation or sense of the arrow. (Note: There is an exception to this approach, namely, the zero vector. The zero vector just sits at the origin, like a point, and has no magnitude and, therefore, an indeterminate direction. This exception will not pose much difficulty.) However, in physics, one doesn't demand that all vectors be represented by arrows having their tails bound to the origin. One is free to "parallel translate" vectors throughout $\mathbf{R}^2$ and $\mathbf{R}^3$. That is, one may represent the vector $\mathbf{a} = (a_1, a_2, a_3)$ by an arrow with its tail at the origin (and its head at $(a_1, a_2, a_3)$) or with its tail at any other point, so long as the length and sense of the arrow are not disturbed. (See Figure 1.4.) For example, if we wish to represent $\mathbf{a}$ by an arrow with its tail at the point $(x_1, x_2, x_3)$, then the head of the arrow would be at the point $(x_1 + a_1, x_2 + a_2, x_3 + a_3)$. (See Figure 1.5.)

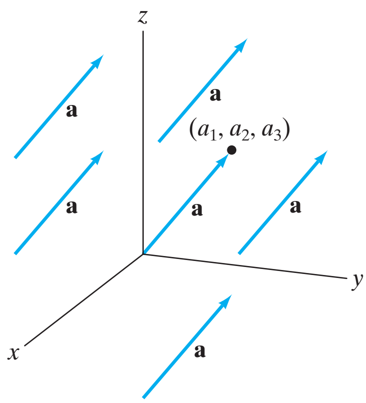



**FIGURE 1.4** Each arrow is a parallel translate of the position vector of the point $(a_1, a_2, a_3)$ and represents the same vector.

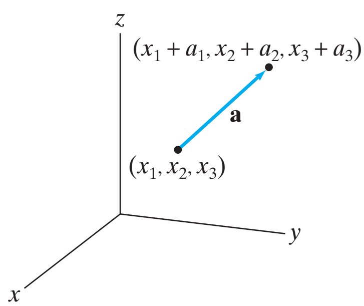



**FIGURE 1.5** The vector $\mathbf{a} = (a_1, a_2, a_3)$ represented by an arrow with tail at the point $(x_1, x_2, x_3)$.

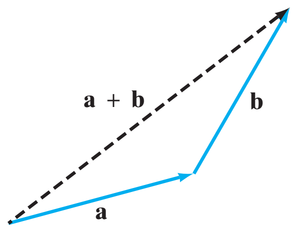


**FIGURE 1.6** The vector $\mathbf{a} + \mathbf{b}$ may be represented by an arrow whose tail is at the tail of **a** and whose head is at the head of **b**.

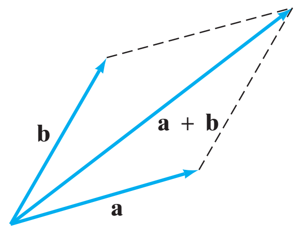


**FIGURE 1.7** The vector $\mathbf{a} + \mathbf{b}$ may be represented by the arrow that runs along the diagonal of the parallelogram determined by **a** and **b**.

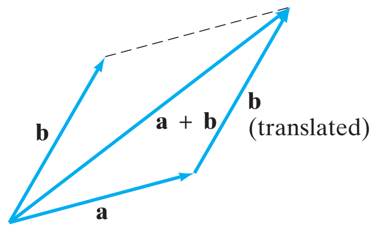


**FIGURE 1.8** The equivalence of the parallelogram law and the head-to-tail methods of vector addition.

With this geometric description of vectors, vector addition can be visualized in two ways. The first is often referred to as the "head-to-tail" method for adding vectors. Draw the two vectors **a** and **b** to be added so that the tail of one of the vectors, say **b**, is at the head of the other. Then the vector sum $\mathbf{a} + \mathbf{b}$ may be represented by an arrow whose tail is at the tail of **a** and whose head is at the head of **b**. (See Figure 1.6.) Note that it is *not* immediately obvious that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ from this construction!

The second way to visualize vector addition is according to the so-called **parallelogram law**: If **a** and **b** are nonparallel vectors drawn with their tails emanating from the same point, then $\mathbf{a} + \mathbf{b}$ may be represented by the arrow (with its tail at the common initial point of **a** and **b**) that runs along a diagonal of the parallelogram determined by **a** and **b** (Figure 1.7). The parallelogram law is completely consistent with the head-to-tail method. To see why, just parallel translate **b** to the opposite side of the parallelogram. Then the diagonal just described is the result of adding **a** and (the translate of) **b**, using the head-to-tail method. (See Figure 1.8.)

We still should check that these geometric constructions agree with our algebraic definition. For simplicity, we'll work in $\mathbf{R}^2$. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ as usual. Then the arrow obtained from the parallelogram law addition of **a** and **b** is the one whose tail is at the origin $O$ and whose head is at the point $P$ in Figure 1.9. If we parallel translate **b** so that its tail is at the head of **a**, then it is immediate that the coordinates of $P$ must be $(a_1 + b_1, a_2 + b_2)$, as desired.

Scalar multiplication is easier to visualize: The vector $k\mathbf{a}$ may be represented by an arrow whose length is $|k|$ times the length of **a** and whose direction is the same as that of **a** when $k > 0$ and the opposite when $k < 0$. (See Figure 1.10.)

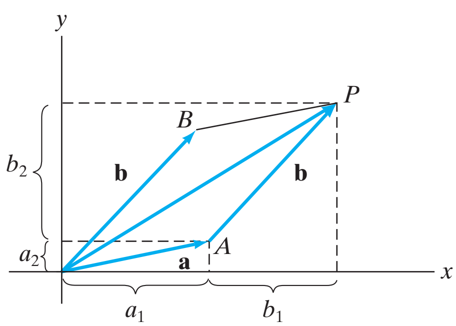


**FIGURE 1.9** The point $P$ has coordinates $(a_1 + b_1, a_2 + b_2)$.

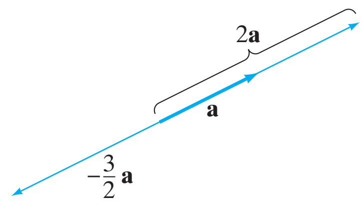


**FIGURE 1.10** Visualization of scalar multiplication.

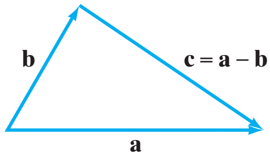



**FIGURE 1.11** The geometry of vector subtraction. The vector $\mathbf{c}$ is such that $\mathbf{b} + \mathbf{c} = \mathbf{a}$. Hence, $\mathbf{c} = \mathbf{a} - \mathbf{b}$.

It is now a simple matter to obtain a geometric depiction of the **difference** between two vectors. (See Figure 1.11.) The difference $\mathbf{a} - \mathbf{b}$ is nothing more than $\mathbf{a} + (-\mathbf{b})$ (where $-\mathbf{b}$ means the scalar $-1$ times the vector $\mathbf{b}$). The vector $\mathbf{a} - \mathbf{b}$ may be represented by an arrow pointing from the head of $\mathbf{b}$ toward the head of $\mathbf{a}$; such an arrow is also a diagonal of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. (As we have seen, the other diagonal can be used to represent $\mathbf{a} + \mathbf{b}$.) Note in Figure 1.11 that adding $\mathbf{b}$ to $\mathbf{c} = \mathbf{a} - \mathbf{b}$ using the "head-to-tail" method results in vector $\mathbf{a}$, precisely as one would expect of $\mathbf{b} + (\mathbf{a} - \mathbf{b})$.

Here is a construction that will be useful to us from time to time.

**DEFINITION 1.5** Given two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ in $\mathbf{R}^3$, the **displacement vector from $P_1$ to $P_2$** is

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

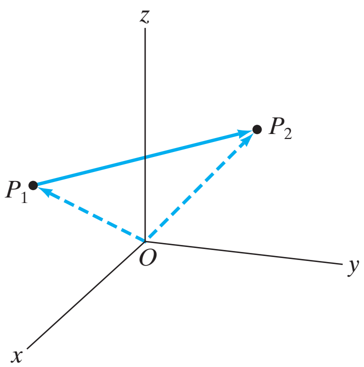



**FIGURE 1.12** The displacement vector $\overrightarrow{P_1P_2}$, represented by the arrow from $P_1$ to $P_2$, is the difference between the position vectors of these two points.

This construction is not hard to understand if we consider Figure 1.12. Given the points $P_1$ and $P_2$, draw the corresponding position vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. Then we see that $\overrightarrow{P_1P_2}$ is precisely $\overrightarrow{OP_2} - \overrightarrow{OP_1}$. An analogous definition may be made for $\mathbf{R}^2$.

In your study of the calculus of one variable, you no doubt used the notions of derivatives and integrals to look at such physical concepts as velocity, acceleration, force, etc. The main drawback of the work you did was that the techniques involved allowed you to study only *rectilinear,* or straight-line, activity. Intuitively, we all understand that motion in the plane or in space is more complicated than straight-line motion. Because vectors possess direction as well as magnitude, they are ideally suited for two- and three-dimensional dynamical problems.

For example, suppose a particle in space is at the point $(a_1, a_2, a_3)$ (with respect to some appropriate coordinate system). Then it has position vector $\mathbf{a} = (a_1, a_2, a_3)$. If the particle travels with constant velocity $\mathbf{v} = (v_1, v_2, v_3)$ for $t$ seconds, then the particle's displacement from its original position is $t\mathbf{v}$, and its new coordinate position is $\mathbf{a} + t\mathbf{v}$. (See Figure 1.13.)

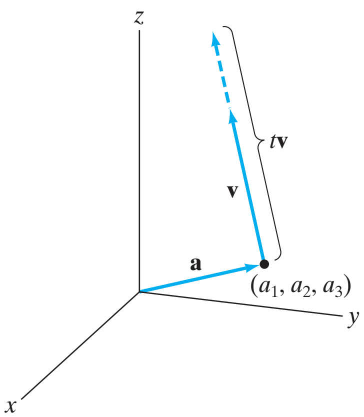



**FIGURE 1.13** After $t$ seconds, the point starting at $\mathbf{a}$, with velocity $\mathbf{v}$, moves to $\mathbf{a} + t\mathbf{v}$.

**EXAMPLE 4** If a spaceship is at position $(100, 3, 700)$ and is traveling with velocity $(7, -10, 25)$ (meaning that the ship travels 7 mi/sec in the positive $x$-direction, 10 mi/sec in the negative $y$-direction, and 25 mi/sec in the positive $z$-direction), then after 20 seconds, the ship will be at position

$$(100, 3, 700) + 20(7, -10, 25) = (240, -197, 1200),$$

and the displacement from the initial position is $(140, -200, 500)$. ▭

**EXAMPLE 5** The S.S. Calculus is cruising due south at a rate of 15 knots (nautical miles per hour) with respect to still water. However, there is also a current of $5\sqrt{2}$ knots southeast. What is the total velocity of the ship? If the ship is initially at the origin and a lobster pot is at position $(20, -79)$, will the ship collide with the lobster pot?

Since velocities are vectors, the total velocity of the ship is $\mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is the velocity of the ship with respect to still water and $\mathbf{v}_2$ is the

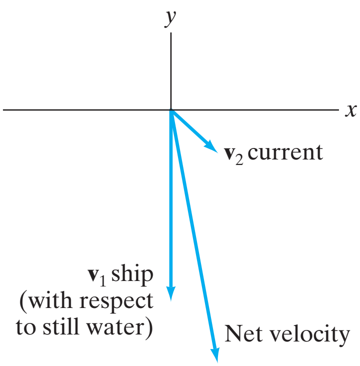



**FIGURE 1.14** The length of $\mathbf{v}_1$ is 15, and the length of $\mathbf{v}_2$ is $5\sqrt{2}$.

southeast-pointing velocity of the current. Figure 1.14 makes it fairly straightforward to compute these velocities. We have that $\mathbf{v}_1 = (0, -15)$. Since $\mathbf{v}_2$ points southeastward, its direction must be along the line $y = -x$. Therefore, $\mathbf{v}_2$ can be written as $\mathbf{v}_2 = (v, -v)$, where $v$ is a positive real number. By the Pythagorean theorem, if the length of $\mathbf{v}_2$ is $5\sqrt{2}$, then we must have $v^2 + (-v)^2 = (5\sqrt{2})^2$ or $2v^2 = 50$, so that $v = 5$. Thus, $\mathbf{v}_2 = (5, -5)$, and, hence, the net velocity is

$$(0, -15) + (5, -5) = (5, -20).$$

After 4 hours, therefore, the ship will be at position

$$(0, 0) + 4(5, -20) = (20, -80)$$

and thus will miss the lobster pot.

**EXAMPLE 6** The theory behind the venerable martial art of judo is an excellent example of vector addition. If two people, one relatively strong and the other relatively weak, have a shoving match, it is clear who will prevail. For example, someone pushing one way with 200 lb of force will certainly succeed in overpowering another pushing the opposite way with 100 lb of force. Indeed, as Figure 1.15 shows, the net force will be 100 lb in the direction in which the stronger person is pushing.


**FIGURE 1.15** A relatively strong person pushing with a force of 200 lb can quickly subdue a relatively weak one pushing with only 100 lb of force.

Dr. Jigoro Kano, the founder of judo, realized (though he never expressed his idea in these terms) that this sort of vector addition favors the strong over the weak. However, if weaker participants apply their 100 lb of force in a direction only slightly different from that of a stronger one, they will effect a vector sum of length large enough to surprise the opponent. (See Figure 1.16.) This is the basis for essentially all of the throws of judo and why judo is described as the art of "using a person's strength against oneself." In fact, the word "judo" means "the way of gentleness" or "the way of giving in." One "gives in" to the strength of another by attempting only to redirect his or her force rather than to oppose it.

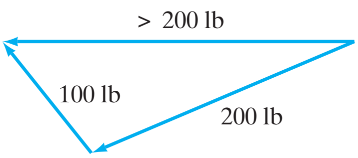



**FIGURE 1.16** Vector addition in judo.

## 1.1 Exercises

**1.** Sketch the following vectors in $\mathbf{R}^2$:

(a) $(2, 1)$

(b) $(3, 3)$

(c) $(-1, 2)$

**2.** Sketch the following vectors in $\mathbf{R}^3$:

(a) $(1, 2, 3)$

(b) $(-2, 0, 2)$

(c) $(2, -3, 1)$

**3.** Perform the indicated algebraic operations. Express your answers in the form of a single vector $\mathbf{a} = (a_1, a_2)$ in $\mathbf{R}^2$.

(a) $(3, 1) + (-1, 7)$

(b) $-2(8, 12)$

(c) $(8, 9) + 3(-1, 2)$

(d) $(1, 1) + 5(2, 6) - 3(10, 2)$

(e) $(8, 10) + 3((8, -2) - 2(4, 5))$

**4.** Perform the indicated algebraic operations. Express your answers in the form of a single vector $\mathbf{a} = (a_1, a_2, a_3)$ in $\mathbf{R}^3$.

(a) $(2, 1, 2) + (-3, 9, 7)$

(b) $\frac{1}{2}(8, 4, 1) + 2(5, -7, \frac{1}{4})$

(c) $-2((2, 0, 1) - 6(\frac{1}{2}, -4, 1))$

**5.** Graph the vectors $\mathbf{a} = (1, 2)$, $\mathbf{b} = (-2, 5)$, and $\mathbf{a} + \mathbf{b} = (1, 2) + (-2, 5)$, using both the parallelogram law and the head-to-tail method.

**6.** Graph the vectors $\mathbf{a} = (3, 2)$ and $\mathbf{b} = (-1, 1)$. Also calculate and graph $\mathbf{a} - \mathbf{b}$, $\frac{1}{2}\mathbf{a}$, and $\mathbf{a} + 2\mathbf{b}$.

**7.** Let $A$ be the point with coordinates $(1, 0, 2)$, let $B$ be the point with coordinates $(-3, 3, 1)$, and let $C$ be the point with coordinates $(2, 1, 5)$.

(a) Describe the vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

(b) Describe the vectors $\overrightarrow{AC}$, $\overrightarrow{BC}$, and $\overrightarrow{AC} + \overrightarrow{CB}$.

(c) Explain, with pictures, why $\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$.

**8.** Graph $(1, 2, 1)$ and $(0, -2, 3)$, and calculate and graph $(1, 2, 1) + (0, -2, 3)$, $-1(1, 2, 1)$, and $4(1, 2, 1)$.

**9.** If $(-12, 9, z) + (x, 7, -3) = (2, y, 5)$, what are $x$, $y$, and $z$?

**10.** What is the length (magnitude) of the vector $(3, 1)$? (Hint: A diagram will help.)

**11.** Sketch the vectors $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (5, 10)$. Explain why $\mathbf{a}$ and $\mathbf{b}$ point in the same direction.

**12.** Sketch the vectors $\mathbf{a} = (2, -7, 8)$ and $\mathbf{b} = (-1, \frac{7}{2}, -4)$. Explain why $\mathbf{a}$ and $\mathbf{b}$ point in opposite directions.

**13.** How would you add the vectors $(1, 2, 3, 4)$ and $(5, -1, 2, 0)$ in $\mathbf{R}^4$? What should $2(7, 6, -3, 1)$ be? In general, suppose that

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad \mathbf{b} = (b_1, b_2, \ldots, b_n)$$

are two vectors in $\mathbf{R}^n$ and $k \in \mathbf{R}$ is a scalar. Then how would you define $\mathbf{a} + \mathbf{b}$ and $k\mathbf{a}$?

**14.** Find the displacement vectors from $P_1$ to $P_2$, where $P_1$ and $P_2$ are the points given. Sketch $P_1$, $P_2$, and $\overrightarrow{P_1P_2}$.

(a) $P_1(1, 0, 2)$, $P_2(2, 1, 7)$

(b) $P_1(1, 6, -1)$, $P_2(0, 4, 2)$

(c) $P_1(0, 4, 2)$, $P_2(1, 6, -1)$

(d) $P_1(3, 1)$, $P_2(2, -1)$

**15.** Let $P_1(2, 5, -1, 6)$ and $P_2(3, 1, -2, 7)$ be two points in $\mathbf{R}^4$. How would you define and calculate the displacement vector from $P_1$ to $P_2$? (See Exercise 13.)

**16.** If $A$ is the point in $\mathbf{R}^3$ with coordinates $(2, 5, -6)$ and the displacement vector from $A$ to a second point $B$ is $(12, -3, 7)$, what are the coordinates of $B$?

**17.** Suppose that you and your friend are in New York talking on cellular phones. You inform each other of your own displacement vectors from the Empire State Building to your current position. Explain how you can use this information to determine the displacement vector from you to your friend.

**18.** Give the details of the proofs of properties 2 and 3 of vector addition given in this section.

**19.** Prove the properties of scalar multiplication given in this section.

**20.** (a) If $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$, what is $0\mathbf{a}$? Prove your answer.

(b) If $\mathbf{a}$ is a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$, what is $1\mathbf{a}$? Prove your answer.

**21.** (a) Let $\mathbf{a} = (2, 0)$ and $\mathbf{b} = (1, 1)$. For $0 \le s \le 1$ and $0 \le t \le 1$, consider the vector $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$. Explain why the vector $\mathbf{x}$ lies in the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$. (Hint: It may help to draw a picture.)

(b) Now suppose that $\mathbf{a} = (2, 2, 1)$ and $\mathbf{b} = (0, 3, 2)$. Describe the set of vectors $\{\mathbf{x} = s\mathbf{a} + t\mathbf{b} \,|\, 0 \le s \le 1, 0 \le t \le 1\}$.

**22.** Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two nonzero vectors such that $\mathbf{b} \ne k\mathbf{a}$. Use vectors to describe the set of points inside the parallelogram with vertex $P_0(x_0, y_0, z_0)$ and whose adjacent sides are parallel to $\mathbf{a}$ and $\mathbf{b}$ and have the same lengths as $\mathbf{a}$ and $\mathbf{b}$. (See Figure 1.17.) (Hint: If $P(x, y, z)$ is a point in the parallelogram, describe $\overrightarrow{OP}$, the position vector of $P$.)

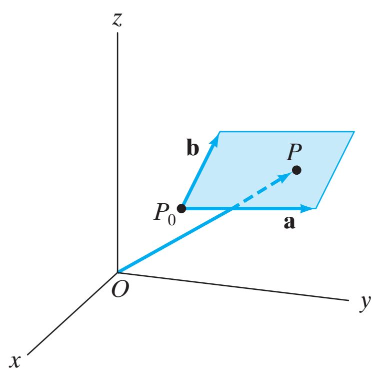



**FIGURE 1.17** Figure for Exercise 22.

**23.** A flea falls onto marked graph paper at the point $(3, 2)$. She begins moving from that point with velocity vector $\mathbf{v} = (-1, -2)$ (i.e., she moves 1 graph paper unit per minute in the negative $x$-direction and 2 graph paper units per minute in the negative $y$-direction).

(a) What is the speed of the flea?

(b) Where is the flea after 3 minutes?

(c) How long does it take the flea to get to the point $(-4, -12)$?

(d) Does the flea reach the point $(-13, -27)$? Why or why not?

**24.** A plane takes off from an airport with velocity vector $(50, 100, 4)$. Assume that the units are miles per hour, that the positive $x$-axis points east, and that the positive $y$-axis points north.

(a) How fast is the plane climbing vertically at take-off?

(b) Suppose the airport is located at the origin and a skyscraper is located 5 miles east and 10 miles north of the airport. The skyscraper is 1,250 feet tall. When will the plane be directly over the building?

(c) When the plane is over the building, how much vertical clearance is there?

**25.** As mentioned in the text, physical forces (e.g., gravity) are quantities possessing both magnitude and direction and therefore can be represented by vectors. If an object has more than one force acting on it, then the **resultant** (or **net**) **force** can be represented by the sum of the individual force vectors. Suppose that two forces, $\mathbf{F}_1 = (2, 7, -1)$ and $\mathbf{F}_2 = (3, -2, 5)$, act on an object.

(a) What is the resultant force of $\mathbf{F}_1$ and $\mathbf{F}_2$?

(b) What force $\mathbf{F}_3$ is needed to counteract these forces (i.e., so that *no* net force results and the object remains at rest)?

**26.** A 50 lb sandbag is suspended by two ropes. Suppose that a three-dimensional coordinate system is introduced so that the sandbag is at the origin and the ropes are anchored at the points $(0, -2, 1)$ and $(0, 2, 1)$.

(a) Assuming that the force due to gravity points parallel to the vector $(0, 0, -1)$, give a vector $\mathbf{F}$ that describes this gravitational force.

(b) Now, use vectors to describe the forces along each of the two ropes. Use symmetry considerations and draw a figure of the situation.

**27.** A 10 lb weight is suspended in equilibrium by two ropes. Assume that the weight is at the point $(1, 2, 3)$ in a three-dimensional coordinate system, where the positive $z$-axis points straight up, perpendicular to the ground, and that the ropes are anchored at the points $(3, 0, 4)$ and $(0, 3, 5)$. Give vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ that describe the forces along the ropes.

## 1.2 More About Vectors

### The Standard Basis Vectors

In $\mathbf{R}^2$, the vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ play a special notational role. Any vector $\mathbf{a} = (a_1, a_2)$ may be written in terms of $\mathbf{i}$ and $\mathbf{j}$ via vector addition and scalar multiplication:

$$(a_1, a_2) = (a_1, 0) + (0, a_2) = a_1(1, 0) + a_2(0, 1) = a_1\mathbf{i} + a_2\mathbf{j}.$$

(It may be easier to follow this argument by reading it in reverse.) Insofar as notation goes, the preceding work simply establishes that one can write either $(a_1, a_2)$ or $a_1\mathbf{i} + a_2\mathbf{j}$ to denote the vector $\mathbf{a}$. It's your choice which notation to use (as long as you're consistent), but the **ij**-notation is generally useful for emphasizing the "vector" nature of **a**, while the coordinate notation is more useful for emphasizing the "point" nature of **a** (in the sense of **a**'s role as a possible position vector of a point). Geometrically, the significance of the **standard basis vectors i** and **j** is that an arbitrary vector $\mathbf{a} \in \mathbf{R}^2$ can be decomposed pictorially into appropriate **vector components** along the $x$- and $y$-axes, as shown in Figure 1.18.

Exactly the same situation occurs in $\mathbf{R}^3$, except that we need three vectors, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, to form the standard basis. (See Figure 1.19.) The same argument as the one just given can be used to show that any vector $\mathbf{a} = (a_1, a_2, a_3)$ may also be written as $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. We shall use both coordinate and standard basis notation throughout this text.

**EXAMPLE 1** We may write the vector $(1, -2)$ as $\mathbf{i} - 2\mathbf{j}$ and the vector $(7, \pi, -3)$ as $7\mathbf{i} + \pi\mathbf{j} - 3\mathbf{k}$.